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GATE
Engineering Mathematics

- *Fully Solved with Explanations*
- *Topicwise Presentation*
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CHAPTER 1

Linear Algebra

1.1 INTRODUCTION

Linear Algebra is a branch of mathematics concerned with the study of vectors, with families of vectors called vector spaces or linear spaces and with functions that input one vector and output another, according to certain rules. These functions are called linear maps or linear transformations and are often represented by matrices. Matrices are rectangular arrays of numbers or symbols and matrix algebra or linear algebra provides the rules defining the operations that can be formed on such an object.

Linear Algebra and matrix theory occupy an important place in modern mathematics and has applications in almost all branches of engineering and physical sciences. An elementary application of linear algebra is to the solution of a system of linear equations in several unknowns, which often result when linear mathematical models are constructed to represent physical problems. Nonlinear models can often be approximated by linear ones. Other applications can be found in computer graphics and in numerical methods.

In this chapter, we shall discuss matrix algebra and its use in solving linear system of algebraic equations $A\hat{x} = b$ and in solving the eigen value problem $A\hat{x} = \lambda\hat{x}$.

1.2 ALGEBRA OF MATRICES

1.2.1 Definition of Matrix

A system of mn numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order $m \times n$.

If $A = [a_{ij}]_{m \times n}$ be any matrix of order $m \times n$ then it is written in the form:

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \cdots & \cdots\cdots\cdots \\ \cdots & \cdots\cdots\cdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

1.2.2 Special Types of Matrices

1. **Square Matrix:** An m × n matrix for which m = n (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. The elements $a_{ij} \mid i = j$, i.e. a_{11}, a_{22}... are called **DIAGONAL ELEMENTS** and the line along which they lie is called **PRINCIPLE DIAGONAL** of matrix. Elements other than a_{11}, a_{22}, etc are called off-diagonal elements i.e. $a_{ij} \mid i \neq j$.

   Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3 \times 3}$ is a square Matrix

   Note: A square sub-matrix of a square matrix A is called a **"principle sub-matrix"** if its diagonal elements are also the diagonal elements of the matrix A. So $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ is a principle sub matrix of the matrix A given above, but $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ is not.

2. **Diagonal Matrix:** A square matrix in which all off-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero.

   Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a diagonal matrix

   The above matrix can also be written as A = diag [3, 5, 9]

   Properties of Diagonal Matrix:

   diag [x, y, z] + diag [p, q, r] = diag [x + p, y + q, z + r]

   diag [x, y, z] × diag [p, q, r] = diag [xp, yq, zr]

   $(\text{diag}\,[x, y, z])^{-1} = \text{diag}\,[1/x, 1/y, 1/z]$

   $(\text{diag}\,[x, y, z])^{t} = \text{diag}\,[x, y, z]$

   $(\text{diag}\,[x, y, z])^{n} = \text{diag}\,[x^n, y^n, z^n]$

   Eigen values of diag [x, y, z] = x, y and z.

   Determinant of diag [x, y, z] = $|\text{diag}\,[x, y, z]|$ = xyz

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal.

   Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix.

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I. Identity matrix is always square.

   Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when i = j and $a_{ij} = 0$ when $i \neq j$.

   Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Properties of Identity Matrix:

(a) I is Identity element for multiplication, so it is called multiplicative identity.

(b) AI = IA = A

(c) $I^n = I$

(d) $I^{-1} = I$

(e) $|I| = 1$

5. **Null Matrix:** The m × n matrix whose elements are all zero is called null matrix. Null matrix is denoted by O. Null matrix need not be square.

Example: $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

Properties of Null Matrix:

(a) A + O = O + A = A

So, O is additive identity.

(b) A + (–A) = O

6. **Upper Triangular Matrix:** An upper triangular matrix is a square matrix whose lower off-diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$

It is denoted by U.

The diagonal and upper off diagonal elements may or may not be zero.

Example: $U = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

7. **Lower Triangular Matrix:** A lower triangular matrix is a square matrix whose upper off-diagonal triangular elements are zero, i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off-diagonal elements may or may not be zero.

It is denoted by L.

Example: $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & 6 \end{bmatrix}$

8. **Idempotent Matrix:** A matrix A is called Idempotent iff $A^2 = A$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are examples of Idempotent matrices.

9. **Involutory Matrix:** A matrix A is called Involutory iff $A^2 = I$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is Involutory. Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is Involutory since $A^2 = I$.

10. **Nilpotent Matrix:** A matrix A is said to be nilpotent of class x or index x iff $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$.

Example: The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since $A \neq 0$ and $A^2 \neq 0$, but $A^3 = 0$.

1.2.3 Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if,

1. They are of same size.
2. The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example: Let $\begin{bmatrix} x-y & p+q \\ p-q & x+y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$

Then $x - y = 2$, $p + q = 5$, $p - q = 1$ and $x + y = 10$

$\Rightarrow \quad x = 6, y = 4, p = 3$ and $q = 2$.

1.2.4 Addition of Matrices

Two matrices A and B are compatible for addition only if they both have exactly the same size say m × n. Then their sum is defined to be the matrix of the type m × n obtained by adding corresponding elements of A and B. Thus if, $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$;

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$$

Properties of Matrix Addition:

1. Matrix addition is commutative $A + B = B + A$.
2. Matrix addition is associative $(A + B) + C = A + (B + C)$
3. Existence of additive identity: If O be m × n matrix each of whose elements are zero. Then, $A + O = A = O + A$ for every m × n matrix A.
4. Existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$.
   Then the negative of matrix A is defined as matrix $[-a_{ij}]_{m \times n}$ and is denoted by –A.
   ⇒ Matrix –A is additive inverse of A. Because $(-A) + A = O = A + (-A)$. Here O is null matrix of order m x n.
5. Cancellation laws holds good in case of addition of matrices, which is X = –A.
   $A + X = B + X \Rightarrow A = B$
   $X + A = X + B \Rightarrow A = B$
6. The equation $A + X = 0$ has a unique solution in the set of all m × n matrices.

1.2.5 Substraction of Two Matrices

If A and B are two m × n matrices, then we define, $A - B = A + (-B)$.

Thus the difference A – B is obtained by subtracting from each element of A corresponding elements of B.

Note: Subtraction of matrices is neither commutative nor associative.

1.2.6 Multiplication of a Matrix by a Scalar

Let A be any m × n matrix and k be any real number called scalar. The m × n matrix obtained by multiplying every element of the matrix A by k is called scalar multiple of A by k and is denoted by kA.

$\Rightarrow$ If $A = [a_{ij}]_{m \times n}$ then $Ak = kA = [kA]_{m \times n}$.

If $A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -5 & 2 \\ 1 & 3 & 6 \end{bmatrix}$ then, $3A = \begin{bmatrix} 15 & 6 & 3 \\ 18 & -15 & 6 \\ 3 & 9 & 18 \end{bmatrix}$

Properties of Multiplication of a Matrix by a Scalar:

1. Scalar multiplication of matrices distributes over the addition of matrices i.e., k(A+B) = kA + kB.
2. If p and q are two scalars and A is any m × n matrix then, (p + q)A = pA + qA.
3. If p and q are two matrices and $A = [a_{ij}]_{m \times n}$ then, p(qA) = (pq)A.
4. If $A = [a_{ij}]_{m \times n}$ be a matrix and k be any scalar then, (−k)A = −(kA) = k(−A).

1.2.7 Multiplication of Two Matrices

Let $A = [a_{ij}]_{m \times n}$; $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [c_{ik}]_{m \times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ is called the product of matrices A and B in that order and we write C = AB.

Properties of Matrix Multiplication:

1. Multiplication of matrices is not commutative. In fact, if the product of AB exists, then it is not necessary that the product of BA will also exist. For example, $A_{3 \times 2} \times B_{2 \times 4} = C_{3 \times 4}$ but $B_{2 \times 4} \times A_{3 \times 2}$ does not exist since these are not compatible for multiplication.
2. Matrix multiplication is associative, if conformability is assured. i.e., A (BC) = (AB)C where A, B, C are m × n, n × p, p × q matrices respectively.
3. Multiplication of matrices is distributive with respect to addition of matrices. i.e., A (B+C) = AB + AC.
4. The equation AB = O does not necessarily imply that at least one of matrices A and B must be a zero matrix. For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
5. In the case of matrix multiplication if AB = O then it is not necessarily imply that BA = O. In fact, BA may not even exist.
6. Both left and right cancellation laws hold for matrix multiplication as shown below:
   AB = AC ⇒ B = C (iff A is non-singular matrix) and
   BA = CA ⇒ B = C (iff A is non-singular matrix).

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 Consider the matrices $X_{(4 \times 3)}$, $Y_{(4 \times 3)}$ and $P_{(2 \times 3)}$. The order of $[P(X^T Y)^{-1} P^T]^T$ will be

(a) (2 × 2) (b) (3 × 3)
(c) (4 × 3) (d) (3 × 4)

[CE, GATE-2005, 1 mark]

Solution: (a)

With the given order we can say that order of matrices are as follows:

$$X^T \rightarrow 3 \times 4$$
$$Y \rightarrow 4 \times 3$$
$$X^TY \rightarrow 3 \times 3$$
$$(X^TY)^{-1} \rightarrow 3 \times 3$$
$$P \rightarrow 2 \times 3$$
$$P^T \rightarrow 3 \times 2$$
$$P(X^TY)^{-1}P^T \rightarrow (2\times3)(3\times3)(3\times2) \rightarrow 2\times2$$
$$\therefore \quad (P(X^TY)^{-1}P^T)^T \rightarrow 2\times2$$

Q.2 There are three matrixes $P(4\times2)$, $Q(2\times4)$ and $R(4\times1)$. The minimum of multiplication required to compute the matrix PQR is

[CE, GATE-2013, 1 Mark]

Solution:

The minimum number of multiplications required to multiply $A_{m\times n}$ with $B_{n\times p}$ is mnp. To compute PQR if we multiply PQ first and then R the number of multiplications required would be $4\times2\times4$ to get PQ and then $4\times4\times1$ multiplications to mulitply PQ with R. So total multiplications required in this method is

$$4\times2\times4+4\times4\times1 = 32+16 = 48$$

To compute PQR if we multiply QR first and then P the number of multiplications required would be $2\times4\times1$ to get QR and then $4\times2\times1$ multiplications to mulitply P with QR. So total multiplications required in this method is

$$2\times4\times1+4\times2\times1 = 8+8 = 16$$

Therfore, the minimum of multiplication required to compute the matrix PQR is = 16

Q.3 Multiplication of matrices E and F is G. Matrices E and G are

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[ME, GATE-2006, 2 marks]

Solution: (c)

Method 1:

$$E = \begin{bmatrix} \cos & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and
$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

According to problem

$$E \times F = G$$

or
$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we see that product of (E × F) is unit matrix so F has to be the inverse of E.

$$F = E^{-1} = \frac{\text{Adj}(E)}{|E|} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2:

An easier method for finding F is by multiplying E with each of the choices (a), (b), (c) and (d) and finding out which one gives the product as identity matrix G. Again the answer is (c).

1.2.8 Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by Tr(A).

Thus if $A = [a_{ij}]_{n \times n}$ then, $\text{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots a_{nn}$.

Let
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

Then, trace (A) = tr(A) = 1+(−3) + 5 = 3

Properties of Trace of a Matrix:

Let A and B be two square matrices of order n and λ be a scalar. Then,

1. tr (λA) = λ tr A
2. tr (A + B) = tr A + tr B
3. tr (AB) = tr (BA)

1.2.9 Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the n × m matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A′ or A^T.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$ then, $A^T = A' = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$

If B = [1 2 3] then

$$B' = [1 \ 2 \ 3]' = [1 \ 2 \ 3]^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of Transpose of a Matrix:

If A′ and B′ be transposes of A and B respectively then,

1. $(A')' = A$
2. $(A + B)' = A' + B'$
3. $(kA)' = kA'$, k being any complex number
4. $(AB)' = B'A'$
5. $(ABC)' = C'\,B'\,A'$

1.2.10 Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ +i & 6 & 9-i \end{bmatrix}$$

Properties of Conjugate of a Matrix:

If $\bar{A}$ & $\bar{B}$ be the conjugates of A & B respectively. Then,

1. $\overline{(\bar{A})} = A$
2. $\overline{(A+B)} = \bar{A}+\bar{B}$
3. $\overline{(kA)} = \bar{k}\bar{A}$, k being any complex number
4. $\overline{(AB)} = \bar{A}\bar{B}$. A & B being conformable to multiplication
5. $\bar{A} = A$ iff A is real matrix
   $\bar{A} = -A$ iff A is purely imaginary matrix

1.2.11 Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ or A^* or $\left(\bar{A}\right)^T$. It is also called conjugate transpose of A.

Example: If $A = \begin{bmatrix} 2+i & 3-i \\ 4 & 1-i \end{bmatrix}$

To find A^θ, we first find $\bar{A} = \begin{bmatrix} 2-i & 3+i \\ 4 & 1+i \end{bmatrix}$

Then $A^\theta = \left(\bar{A}\right)^T = \begin{bmatrix} 2-i & 4 \\ 3+i & 1+i \end{bmatrix}$

Some properties: If A^θ & B^θ be the transposed conjugates of A and B respectively then,

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta = A^\theta + B^\theta$
3. $(kA)^\theta = \bar{k}A^\theta$, k → complex number
4. $(AB)^\theta = B^\theta A^\theta$

1.2.12 Classification of Real Matrices

Real matrices can be classified into the following three types based on the relationship between A^T and A.

1. Symmetric Matrices ($A^T = A$)
2. Skew Symmetric Matrices ($A^T = -A$)
3. Orthogonal Matrices ($A^T = A^{-1}$ or $AA^T = I$)

1. **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ elements is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i & j.
   In a symmetric matrix, $A^T = A$

   Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix, since $A^T = A$.

   Note: For any matrix A,
   (a) AA^t is always a symmetric matrix.
   (b) $\dfrac{A + A^t}{2}$ is always symmetric matrix.

   Note: If A and B an symmetric, then
   (a) A + B and A – B are also symmetric.
   (b) AB, BA may or may not be symmetric.

2. **Skew Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A if $a_{ij} = -a_{ji}\ \forall\ i, j$.
   In a skew symmetric matrix $A^T = -A$.
   A skew symmetric matrix must have all 0's in the diagonal.

   Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

   Note: For any matrix A, the matrix $\dfrac{A - A^t}{2}$ is always skew symmetric.

3. **Orthogonal Matrix:** A square matrix A is said be orthogonal if:
   $A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if, $AA^T = I = A^TA$.
   Example: The identity matrix is orthogonal since $I^t = I^{-1} = I$.
   Note: Since for an orthogonal matrix A,

$$
\begin{aligned}
AA^T &= I \\
\Rightarrow \quad |AA^T| &= |I| = 1 \\
\Rightarrow \quad |A|\,|A^T| &= 1 \\
\Rightarrow \quad (|A|)^2 &= 1 \\
\Rightarrow \quad |A| &= \pm 1
\end{aligned}
$$

   So the determinant of an orthogonal matrix always has a modulus of 1.

1.2.13 Classification of Complex Matrices

Complex matrices can be classified into the following three types based on relationship between A^θ and A.

1. Hermitian Matrix ($A^\theta = A$)
2. Skew-Hermitian Matrix ($A^\theta = -A$)
3. Unitary Matrix ($A^\theta = A^{-1}$ or $AA^\theta = I$)

1. **Hermitian Matrix:** A necessary and sufficient condition for a matrix A to be Hermitian is that $A^\theta = A$.

   Example: $A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$ is a Hermitian matrix.

2. **Skew-Hermitian Matrix:** A necessary and sufficient condition for a matrix to be skew-Hermitian if $A^\theta = -A$.

   Example: $A = \begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}$ is skew-Hermitian.

3. **Unitary Matrix:** A square matrix A is said to be unitary iff:

$$A^\theta = A^{-1}$$

   Multiplying both sides by A, we get an alternate definition of unitary matrix as given below:
   A square matrix A is said to be unitary iff:

$$AA^\theta = I = A^\theta A$$

   Example: $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is an example of a unitary matrix.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.4 Real matrices $[A]_{3\times1}$, $[B]_{3\times3}$, $[C]_{3\times5}$, $[D]_{5\times3}$, $[E]_{5\times5}$ and $[F]_{5\times1}$ are given. Matrices [B] and [E] are symmetric.

Following statements are made with respect to these matrices.

1. Matrix product $[F]^T[C]^T[B][C][F]$ is a scalar.
2. Matrix product $[D]^T[F][D]$ is always symmetric.

With reference to above statements, which of the following applies?

(a) Statement 1 is true but 2 is false
(b) Statement 1 is false but 2 is true
(c) Both the statements are true
(d) Both the statements are false

[CE, GATE-2004, 1 mark]

Solution: (a)

Statement 1 is true as shown below.

$[F]^T$ has a size 1×5

$[C]^T$ has a size 5×3

[B] has a size 3×3

[C] has a size 3×5

[F] has a size 5×1

So $[F]^T[C]^T[B][C][F]$ has a size 1×1. Therefore it is a scalar.

So, Statement 1 is true.

Consider Statement 2: $D^T F D$ is always symmetric.
Now $D' F D$ does not exist since $D'_{3\times5}$, $F_{5\times1}$ and $D_{5\times3}$ are not compatible for multiplication since, $D'_{3\times5} F_{5\times1} = X_{3\times1}$ and $X_{3\times1}, D_{5\times3}$ does not exist.
So, Statement 2 is false.

Q.5 [A] is square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$, respectively. Which of the following statements is TRUE?
(a) Both [S] and [D] are symmetric
(b) Both [S] and [D] are skew-symmetric
(c) [S] is skew-symmetric and [D] is symmetric
(d) [S] is symmetric and [D] is skew-symmetric

[CE, GATE-2007, 1 mark]

Solution: (d)

Since $S^t = (A + A^t)^t = A^t + (A^t)^t$
$= A^t + A = S$

i.e. $S^t = S$

$\therefore$ S is symmetric

Since $D^t = (A - A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t) = -D$

i.e. $D^t = -D$

So D is Skew-Symmetric.

Q.6 A square matrix B is skew-symmetric if
(a) $B^T = -B$
(b) $B^T = B$
(c) $B^{-1} = B$
(d) $B^{-1} = B^T$

[CE, GATE-2009, 1 mark]

Solution: (a)

A square matrix B is defined as skew-symmetric if and only if $B^T = -B$, by definition.

Q.7 Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and

$K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product $K^T JK$ is ____.

[CE, GATE-2014 : 1 Mark, Set-1]

Solution:

$$J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$K^T JK = [1\ 2\ -1]\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

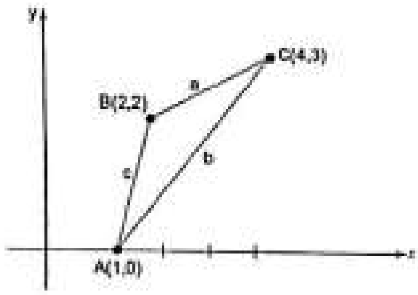
$$= [6\ 8\ -1]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 6+16+1 = 23$$

Q.8 With reference to the conventional Cartesian (x, y) coordinate system, the vertices of a triangle have the following coordinates; $(x_1, y_1) = (1, 0)$; $(x_2, y_2) = (2, 2)$; $(x_3, y_3) = (4, 3)$. The area of the triangle is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$

(c) $\frac{4}{5}$ (d) $\frac{5}{2}$ [CE, GATE-2014 : 1 Mark, Set-I]

Solution : (a)



Area of the triangle $= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$

$$= \frac{1}{2}\left|1(2-3) + 2(3-0) + 4(0-2)\right| = \frac{1}{2}\left|-1+6-8\right| = \frac{3}{2}$$

Q.9 Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I	List-II
A. Singular matrix	1. Determinant is not defined
B. Non-square matrix	2. Determinant is always one
C. Real symmetric	3. Determinant is zero
D. Orthogonal matrix	4. Eigenvalues are always real
	5. Eigenvalues are not defined

Codes:

	A	B	C	D
(a)	3	1	4	2
(b)	2	3	4	1
(c)	3	2	5	4
(d)	3	4	2	1

[ME, GATE-2006, 2 marks]

Solution: (a)

A. Singular matrix → Determinant is zero
B. Non-square matrix → Determinant is not defined
C. Real symmetric → Eigen values are always real
D. Orthogonal matrix → Determinant is always one

Q.10 Which one of the following equations is a correct identity for arbitrary 3 × 3 real matrices P, Q and R?

(a) $P(Q + R) = PQ + RP$
(b) $(P - Q)^2 = P^2 - 2PQ + Q^2$
(c) $\det(P + Q) = \det P + \det Q$
(d) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (d)

$$(P + Q)^2 = P^2 + PQ + QP + Q^2 = P.P + P.Q + Q.P + Q.Q = P^2 + PQ + QP + Q^2$$

Q.11 Which one of the following statements is true for all real symmetric matrices?

(a) All the eigenvalues are real
(b) All the eigenvalues are positive.
(c) All the eigenvalues are distinct
(d) Sum of all the eigenvalues is zero.

[EE, GATE-2014 : 1 Mark, Set-2]

Answer : (a)

Q.12 Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $\left[AA^T\right]^{-1}$ is

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Solution: (c)

For orthogonal matrix $AA^T = I$ i.e. Identity matrix.

$\therefore \quad (AA^T)^{-1} = I^{-1} = I$

Q.13 For matrices of same dimension M, N and scalar c, which one of these properties DOES NOT ALWAYS hold?

(a) $(M^T)^T = M$
(b) $(cM)^T = c(M)^T$
(c) $(M + N)^T = M^T + N^T$
(d) $MN = NM$

[EC, GATE-2014 : 1 Mark, Set-1]

Solution : (d)

Matrix multiplication is not commutative.

Q.14 Which one of the following statements is NOT true for a square matrix A?
(a) If A is upper triangular, the eigenvalues of A are the diagonal elements of it
(b) If A is real symmetric, the eigenvalues of A are always real and positive
(c) If A is real, the eigenvalues of A and A^T are always the same
(d) If all the principal minors of A are positive, all the eigenvalues of A are also positive

[EC, GATE-2014 : 2 Marks, Set-3]

Answer : (b)

Q.15 A real square matrix A is called skew-symmetric if
(a) $A^T = A$ (b) $A^T = A^{-1}$
(c) $A^T = -A$ (d) $A^T = A + A^{-1}$

[ME, 2016 : 1 Mark, Set-3]

Solution: (c)
A is skew-symmetric
$\therefore \quad A^T = -A$

Q.16 Let $M^4 = I$, (where I denotes the identity matrix) and $M \neq I$, $M^2 \neq I$ and $M^3 \neq I$. Then, for any natural number k, M^{-1} equals:
(a) M^{4k+1} (b) M^{4k+2}
(c) M^{4k+3} (d) M^{4k}

[EC, 2016 : 1 Mark, Set-1]

Solution: (c)
Given that $M^4 = I$ or $M^{4k} = I$ or $M^{4(k+1)} = I$

$$\therefore \quad M^{-1} \times I = M^{4(k+1)} \times M^{-1}$$

$$\therefore \quad M^{-1} = M^{4k+3}$$

1.3 DETERMINANTS

1.3.1 Definition

Let a_{11}, a_{12}, a_{21}, a_{22} be any four numbers. The symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}a_{22} - a_{12}a_{21}$ and is called determinants of order 2. The number a_{11}, a_{12}, a_{21}, a_{22} are called elements of the determinant and the number $a_{11}a_{22} - a_{21}a_{12}$ is called the value of determinant.

1.3.2 Minors and Cofactors

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Leaving the row and column passing through the elements a_{ij}, then the second order determinant thus obtained is called the minor of element a_{ij} and we will be denoted by M_{ij}.

Example: The Minor of element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

Similarly Minor of element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$

1.3.3 Cofactors

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of element a_{ij}. We shall denote the cofactor of an element by corresponding capital letter.

Example: Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$.

Cofactor of element $a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

by cofactor of element $a_{32} = A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

We define for any matrix, the sum of the products of the elements of any row or column with corresponding cofactors is equal to the determinant of the matrix.

Example: If

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

then,

$$\text{cof}(A) = \begin{bmatrix} 12 & 4 & -12 \\ -4 & 2 & 4 \\ 2 & -1 & 8 \end{bmatrix}$$

$$\begin{aligned} |A| &= (1 \times 12) + (2 \times 4) + (0 \times -12) \\ &= (-1 \times -4) + (6 \times 2) + (1 \times 4) \\ &= (2 \times 2) + (0 \times -1) + (2 \times 8) = 20 \end{aligned}$$

1.3.4 Determinant of order n

A determinant of order n has n-row and n-columns. It has $n \times n$ elements.

A determinant of order n is a square array of $n \times n$ quantities enclosed between vertical bars.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & \cdots\cdots\cdots & a_{2n} \\ a_{n1} & a_{n2} \cdots\cdots & a_{nn} \end{vmatrix}$$

Cofactor of A_{ij} of elements a_{ij} in D is equal to $(-1)^{i+j}$ times the determinants of order $(n-1)$ obtained from D by leaving the row and column passing through element a_{ij}.

If A is a 3×3 matrix, then $|A| = \sum_{j=1}^{3} A_{1j}\,\text{cof}(A_{1j}) = \sum_{j=1}^{3} A_{2j}\,\text{cof}(A_{2j}) = \sum_{j=1}^{3} A_{3j}\,\text{cof}(A_{3j}) = \sum_{i=1}^{3} A_{i1}\,\text{cof}(A_{i1})$, etc.

Therefore, determinant can be expanded using any row or column.

1.3.5 Properties of Determinants

1. The value of a determinant does not change when rows and columns are interchanged. i.e. $|A^T| = |A|$
2. If any row (or column) of a matrix A is completely zero, then $|A| = 0$.
   Such a row (or column) is called a zero row (or column).
   Also if any two rows (or columns) of a matrix A are identical, then $|A| = 0$.
3. If any two rows or two columns of a determinant are interchanged the value of determinant is multiplied by -1.
4. If all elements of the one row (or one column) of a determinant are multiplied by same number k the value of determinant is k times the value of given determinant.
5. If A be n-rowed square matrix, and k be any scalar, then $|kA| = k^n|A|$

6. (a) In a determinant the sum of the products of the elements of any row (or column) with the cofactors of corresponding elements of any row or column is equal to the determinant value.
   (b) In determinant the sum of the products of the elements of any row (or column) with the cofactors of some other row or column is zero.

Example:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then
$$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$$
$$a_1A_2 + b_1B_2 + c_1C_2 = 0$$
$$a_1A_3 + b_1B_3 + c_1C_3 = 0$$
$$a_2A_2 + b_2B_2 + c_2C_2 = \Delta$$
$$a_2A_1 + b_2B_1 + c_2C_1 = 0 \text{ etc}$$

where A_1, B_1, C_1 etc., be cofactors of the elements a_1, b_1, c_1 in D.

7. If to the elements of a row (or column) of a determinant are added m times the corresponding elements of another row (or column) the value of determinant thus obtained is equal to the value of original determinant.

   i.e., $A \xrightarrow{R_i + kR_j} B$ then $|A| = |B|$

   and $A \xrightarrow{C_i + kC_j} B$ then $|A| = |B|$

8. $|AB| = |A| * |B|$ and based on this we can prove the following:
   (a) $|A^n| = (|A|)^n$
   (b) $|A^{-1}| = \dfrac{1}{|A|}$

   Proof of a:
   $$|A^n| = |A * A * A \ldots n \text{ times}|$$
   $$= |A| * |A| * |A| \ldots n \text{ times}$$
   $$= (|A|)^n$$

   Proof of b:
   $$|A A^{-1}| = |I|$$
   $$= 1$$

   Now since, $|A A^{-1}| = |A|\ |A^{-1}|$

   $\therefore$ $|A|\ |A^{-1}| = 1$

   $\Rightarrow$ $|A^{-1}| = \dfrac{1}{|A|}$

9. Using the fact that $A \,.\, \text{Adj } A = |A| \,.\, I$, the following can be proved for $A_{n \times n}$.
   (a) $|\text{Adj } A| = |A|^{n-1}$
   (b) $|\text{Adj } (\text{Adj } (A))| = |A|^{(n-1)^2}$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.17 If any two columns of a determinant $P = \begin{vmatrix} 4 & 7 & 8 \\ 3 & 1 & 5 \\ 9 & 6 & 2 \end{vmatrix}$ are interchanged, which one of the following statements regarding the value of the determinant is CORRECT?

(a) Absolute value remains unchanged but sign will change
(b) Both absolute value and sign will change
(c) Absolute value will change but sign will not change
(d) Both absolute value and sign will remain unchanged

[ME, GATE-2015 : 1 Mark, Set-1]

Solution: (a)

Property of determinant : If any two row or column are interchanged, then mangnitude of determinant remains same but sign changes.

Q.18 Perform the following operations on the matrix $\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$.

1. Add the third row to the second row.
2. Subtract the third column from the first column.

The determinant of the resultant matrix is ______.

[CS, GATE-2015 : 2 Marks, Set-2]

Solution: (0)

Since operations 1 and 2 are elementary operations of the type of $R_i \pm kR_j$ and $C_i \pm kC_j$ respectively, the determinant will be unchanged from the original determinant.

So the required determinant = $\begin{vmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{vmatrix}$

$$\begin{vmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{vmatrix} \xrightarrow{C_3 - 15C_1} \begin{vmatrix} 3 & 4 & 0 \\ 7 & 9 & 0 \\ 13 & 2 & 0 \end{vmatrix} = 0$$

So the required determinant = 0.

Q.19 For $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, the determinant of $A^T A^{-1}$ is

(a) $\sec^2 x$
(b) $\cos 4x$
(c) 1
(d) 0

[EC, GATE-2015 : 1 Mark, Set-3]

Solution: (c)

Long Method:

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}[adj\,(A)]^T = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

Here,

$$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1-\tan^2 x & -2\tan x \\ 2\tan x & 1-\tan^2 x \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} \frac{1-\tan^2 x}{\sec^2 x} & \frac{-2\tan x}{\sec^2 x} \\ \frac{2\tan x}{\sec^2 x} & \frac{1-\tan^2 x}{\sec^2 x} \end{bmatrix}$$

$$|A^T A^{-1}| = \left(\frac{1-\tan^2 x}{\sec^2 x}\right)^2 + \left(\frac{2\tan x}{\sec^2 x}\right)^2$$

$$= \frac{1+\tan^4 x - 2\tan^2 x + 4\tan^2 x}{\sec^4 x} = 1$$

(or)

Short Method:

Since

$$|AB| = |A|\,|B|$$

$$|A^T A^{-1}| = |A^T||A^{-1}|$$

$$= |A| \times \frac{1}{|A|} = 1 \qquad \left(\text{Note}: |A^T| = |A| \text{ and } |A^{-1}| = \frac{1}{|A|}\right)$$

1.4 INVERSE OF MATRIX

The inverse of a matrix A, exists iff A is non-singular (i.e. $|A| \neq 0$) and is given by the formula

$$A^{-1} = \frac{\text{Adj (A)}}{|A|}.$$

Inverse of A when it exists is unique.

1.4.1 Adjoint of a Square Matrix

Let $A=[a_{ij}]$ be any $n \times n$ matrix. The transpose B of the matrix $B = [A_{ij}]_{n \times n}$ where A_{ij} denotes the cofactor of element a_{ij} is called the adjoint of matrix A and is denoted by symbol Adj A.

$\therefore$ $$\text{Adj (A)} = [\text{cof (A)}]^t$$

Properties of Adjoint:

If A be any n-rowed square matrix, then (Adj A) A = A (Adj A) = $|A|\, I_n$

where I_n is the $n \times n$ identity matrix.

1.4.2 Properties of Inverse

1. $AA^{-1} = A^{-1}A = I$
2. A and B are inverse of each other iff AB = BA = I
3. $(AB)^{-1} = B^{-1}A^{-1}$

4. $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$
5. If A be an $n \times n$ non-singular matrix, then $(A')^{-1} = (A^{-1})'$.
6. If A be an $n \times n$ non-singular matrix then $(A^{-1})^{\theta} = (A^{\theta})^{-1}$.
7. For a 2×2 matrix there is a short-cut formula for inverse as given below

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.20 The inverse of the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is

(a) $\frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$

(b) $\frac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$

(d) $\frac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

[CE, GATE-2007, 2 marks]

Solution: (a)

Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix} = \frac{1}{(7-10)} \begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$$

Q.21 The product of matrices $(PQ)^{-1}P$ is

(a) P^{-1}

(b) Q^{-1}

(c) $P^{-1} Q^{-1} P$

(d) $PQ P^{-1}$

[CE, GATE-2008, 1 mark]

Solution: (b)

$$\begin{aligned}(PQ)^{-1} P &= (Q^{-1}P^{-1})P \\ &= (Q^{-1})(P^{-1}P) \\ &= (Q^{-1})(I) \\ &= Q^{-1}\end{aligned}$$

Q.22 The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is

(a) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$

(b) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(c) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$

(d) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

[CE, GATE-2010, 2 marks]

Solution: (b)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}^{-1} = \frac{1}{[(3+2i)(3-2i)+i^2]}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

Q.23 The determinant of matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is ____

[CE, GATE-2014 : 1 Mark, Set-2]

Solution :

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$$

$R_4 \to R_4 - R_2 - R_3$

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & -3 & -2 & 1 \end{vmatrix}$$

$R_4 \to R_4 + 3R_1$

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 4 & 10 \end{vmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 0 & -6 & -8 \\ 0 & 0 & 4 & 10 \end{vmatrix}$$

Interchanging column 1 and column 2 and taking transpose

$$\Delta = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 3 & -6 & 4 \\ 3 & 0 & -8 & 10 \end{vmatrix} = -1\times\begin{vmatrix} 1 & 2 & 0 \\ 3 & -6 & 4 \\ 0 & -8 & 10 \end{vmatrix}$$

$$= -1\times[1(-60+32)+2(0-30)]$$

$$= -(-28-60) = 88$$

Q.24 For which value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

(a) 4
(b) 6
(c) 8
(d) 12

[ME, GATE-2004, 2 marks]

Solution: (a)

$$\text{For singularity of matrix} = \begin{vmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad 8(0-12) - x(0 - 2 \times 12) = 0$$

$$\therefore \quad x = 4$$

Q.25 For a matrix $[M] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix, $[M]^T = [M]^{-1}$. The value of x is given by

(a) $-\frac{4}{5}$
(b) $-\frac{3}{5}$
(c) $\frac{3}{5}$
(d) $\frac{4}{5}$

[ME, GATE-2009, 1 mark]

Solution: (a)

Given $M^T = M^{-1}$.

So $M^TM = I$

$$\Rightarrow \quad \begin{bmatrix} \frac{3}{5} & x \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \left(\frac{3}{5}\right)^2 + x^2 & \left(\frac{3}{5}.\frac{4}{5}\right) + \frac{3}{5}x \\ \left(\frac{4}{5}.\frac{3}{5}\right) + \frac{3}{5}.x & \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Compare both sides a_{12}

$$a_{12} = \left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)x = 0$$

$$\Rightarrow \quad \frac{3}{5}x = -\frac{3}{5}.\frac{4}{5}$$

$$\Rightarrow \quad x = -\frac{4}{5}$$

Q.26 Given that the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is – 12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is

(a) – 96
(b) – 24
(c) 24
(d) 96

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (a)

Let D = – 12 for the given matrix

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix} = (2)^3 \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$$

(Taking 2 common from each row)

$$\therefore \quad \text{Det}(A) = (2)^3 \times D = 8 \times -12 = -96$$

Q.27 For given matrix $P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$ where $i = \sqrt{-1}$, the inverse of matrix P is

(a) $\frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$
(b) $\frac{1}{25}\begin{bmatrix} i & 4-i \\ 4+3i & -i \end{bmatrix}$
(c) $\frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$
(d) $\frac{1}{25}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

[ME, GATE-2015 : 2 Marks, Set-3]

Solution: (a)

$$P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$$

$$P^{-1} = \frac{\begin{bmatrix} 4-3i & -(-i) \\ -i & 4+3i \end{bmatrix}}{|A|} = \frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$$

Q.28 If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

(a) [5 6 4]
(b) [5 –3 1]
(c) [2 0 –1]
(d) [2 –1 1/2]

[EE, GATE-2005, 2 marks]

Solution: (b)

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$R^{-1} = \frac{\text{adj}(R)}{|R|} = \frac{[\text{cofactor }(R)]^T}{|R|}$$

$$|R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 1(2+3) - 0(4+2) - 1(6-2) = 5 - 4 = 1$$

Since we need only the top row of R^{-1}, we need to find only first column of cof (R) which after transpose will become first row of adj (R).

$$\text{cof. }(1, 1) = +\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5$$

$$\text{cof. }(2, 1) = -\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = -3$$

$$\text{cof. }(3, 1) = +\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = +1$$

$$\therefore \quad \text{cof. }(A) = \begin{bmatrix} 5 & - & - \\ -3 & - & - \\ 1 & - & - \end{bmatrix}$$

$$\text{Adj }(A) = [\text{cof. }(A)]^T = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

Dividing by $|R| = 1$ gives

$$R^{-1} = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$\therefore$ top row of $R^{-1} = [5 \ -3 \ 1]$

Q.29 A is $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A' = (A^TA)^{-1}A^T$. Then, which one of the following statement is TRUE?

(a) $AA'A = A$
(b) $(AA')^2 = A$
(c) $AA'A = I$
(d) $AA'A = A'$

[EE, GATE-2008, 2 marks]

Solution: (a)

Choice (a) $AA'A = A$ is correct

Since, $AA'A = A[(A^TA)^{-1}A^T]A = A[(A^TA)^{-1}A^TA]$

Let $A^TA = P$

Then $= A[P^{-1}P] = A.I = A$

Q.30 Let, $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$. Then $(a+b) =$

(a) $\frac{7}{20}$
(b) $\frac{3}{20}$
(c) $\frac{19}{60}$
(d) $\frac{11}{20}$

[EC, GATE-2005, 2 marks]

Solution: (a)

$$[AA^{-1}] = I$$

$$\Rightarrow \quad \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} ½ & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2a-0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2a - 0.1b = 0 \Rightarrow a = 0.1b/2 \quad \text{...(i)}$$

$$3b = 1 \Rightarrow b = \frac{1}{3}$$

Now substitute b in equation (i), we get

$$a = \frac{1}{60}$$

So, $a + b = \frac{1}{60} + \frac{1}{3} = \frac{1+20}{60} = \frac{21}{60} = \frac{7}{20}$

Q.31 Let A be an $m \times n$ matrix and B an $n \times m$ matrix. It is given that determinant $(I_m + AB)$ = determinant $(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(a) 2
(b) 5
(c) 8
(d) 16

[EC, GATE-2013, 2 Marks]

Solution: (b)

Take the determinant of given matrix

$$2[2(4-1) - 1(2-1) + 1(1-2)] - 1[1(4-1) - 1(2-1) + 1(1-2)] + 1[1(2-1) - 2(2-1) + 1(1-1)] - 1[1(1-2) - 2(1-2) + 1(1-1)] = 2[6-1-1] - 1[3-1-1] + 1[1-2+0] - 1[-1+2+0]$$

$$= 2(4) - 1(1) + 1(-1) - 1(1) = 8 - 1 - 1 - 1$$

$$= 5$$

Q.32 The determinant of matrix A is 5 and the determinant of matrix B is 40. The determinant of matrix AB is

[EC, GATE-2014 : 1 Mark, Set-2]

Solution :

$$\text{Determinant of A} = 5$$

$$\text{Determinant of B} = 40$$

$$\text{Determinant of AB} = |A|\,|B|$$

$$= 5 \times 40 = 200$$

Q.33 The maximum value of the determinant among all 2×2 real symmetric matrices with trace 14 is ________.

[EC, GATE-2014 : 2 Marks, Set-2]

Solution :

Let, $$A = \begin{bmatrix} a & x \\ x & b \end{bmatrix}$$

$\Rightarrow$ $$|A| = ab - x^2$$

Given trace (A) $= a + b = 14$

So, $$|A| = a(14-a) - x^2$$

Since, x^2 is always positive maximum value of $a(14-a) - x^2$ occurs only when $x = 0$.

So now, $$|A| = a(14-a) = 14a - a^2.$$

Now maximizing this with respect to a,

$$\frac{d|A|}{da} = 14 - 2a = 0$$

$\Rightarrow$ $$a = 7$$

Since $$\left.\frac{d^2|A|}{da^2}\right|_{a=7} = -2 < 0$$

At $a = 7$, we have a maximum. The maximum value is $14 \times 7 - 7^2 = 49$

Q.34 Let A, B, C, D be $n \times n$ matrices, each with non-zero determinant. If $ABCD = I$, then B^{-1} is

(a) $D^{-1}C^{-1}A^{-1}$
(b) CDA
(c) ADC
(d) does not necessarily exist

[CS, GATE-2004, 1 mark]

Solution: (b)

A, B, C, D is $n \times n$ matrix.

Given $$ABCD = I$$

$\Rightarrow$ $$ABCDD^{-1}C^{-1} = D^{-1}C^{-1}$$

$\Rightarrow$ $$AB = D^{-1}C^{-1}$$

$\Rightarrow$ $$A^{-1}AB = A^{-1}D^{-1}C^{-1}$$

$\Rightarrow$ $$B = A^{-1}D^{-1}C^{-1}$$

$$B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$$
$$= (C^{-1})^{-1} \cdot (D^{-1})^{-1} \cdot (A^{-1})^{-1}$$
$$= CDA$$

Q.35 Which one of the following does NOT equal $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$?

(a) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

(d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

[CS, GATE-2013, 1 Mark]

Solution: (a)

The given matrix can be transformed into the matrix given in options (b)(c) and (d) by elementry operations of the type of $R_i \pm kR_j$ or $C_i \pm kC_j$ only as shown below:

Option (b):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[C_3+C_1]{C_2+C_1} \begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$$

Option (c):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[R_2-R_3]{R_1-R_2} \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Option (d):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[R_2+R_3]{R_1+R_2} \begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Option (a): We can show the given matrix can not be converted into option (a) without doing a column exchange which will change the sign of the determinant as can be seen below:

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[C_3+C_2]{C_2+C_1} \begin{vmatrix} 1 & x+1 & x(x+1) \\ 1 & y+1 & y(y+1) \\ 1 & z+1 & z(z+1) \end{vmatrix} = -\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$$

Q.36 If the matrix A is such that

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} [1 \ 9 \ 5]$$

then the determinant of A is equal to ____.

[CS, GATE-2014 : 1 Mark, Set-2]

Solution :

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} [1 \ 9 \ 5]$$

$$A = \begin{bmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{bmatrix} \Rightarrow |A| = 0$$

Q.37 The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has det(A)=100 and trace (A)=14. The value of $|a-b|$ is ____

[EC, 2016 : 2 Marks, Set-2]

Solution:

$$\text{Trace of } A = 14$$
$$a + 5 + 2 + b = 14$$
$$a + b = 7 \quad \text{...(i)}$$

$$\det(A) = 100$$
$$5\begin{vmatrix} a & 3 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & b \end{vmatrix} = 100$$
$$5 \times 2 \times a \times b = 100$$
$$10\,ab = 100$$
$$ab = 10 \quad \text{... (ii)}$$

From equation (i) and (ii)

either $a = 5, \quad b = 2$

or $a = 2 \quad b = 5$

$$|a - b| = |5 - 2| = 3$$

1.5 RANK OF A MATRIX

Rank is defined for any matrix $A_{m \times n}$ (need not be square)

Some important concepts:

1. **Submatrix of a Matrix:** Suppose A is any matrix of the type m × n. Then a matrix obtained by leaving some rows and some columns from A is called sub-matrix of A.
2. **Rank of a Matrix:** A number r is said to be the rank of a matrix A, if it possesses the following properties:
   (a) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.
   (b) If the matrix A contains any square sub-matrix of order (r + 1) and above, then the determinant of such a matrix should be zero.

   Put together property (a) and (b) give the definition of the rank of a matrix as the "size of the largest non-zero minor".

   Note:

   (a) The rank of a matrix is ≤ r, if all (r + 1) - rowed minors of the matrix vanish.

   (b) The rank of a matrix is ≥ r, if there is at least one r-rowed minor of the matrix which is not equal to zero.

   (c) The rank of transpose of a matrix is same as that of original matrix. i.e. $r(A^T) = r(A)$.

   (d) Rank of a matrix is same as the number of linearly independent row vectors in the matrix as well as the number of linearly independent column vectors in the matrix.

   (e) For any matrix A, rank (A) ≤ min(m,n)
   i.e., maximum rank of $A_{m \times n}$ = min(m, n)

   (f) Rank (AB) ≤ Rank A
   Rank (AB) ≤ Rank B
   So, Rank (AB) ≤ min(Rank A, Rank B)

   (g) Rank (A^t) = Rank (A)

   (h) Rank of a matrix is the number of non-zero rows in its echelon form.

Echelon form: A matrix is in echelon form if only if

1. Leading non-zero element in every row is behind leading non-zero element in previous row. This means below the leading non-zero element in every row all the elements must be zero.
2. All the zero rows should be below all the non-zero rows.

This definition gives an alternate way of calculating the rank of larger matrices (larger than 3 × 3) more easily. To reduce a matrix to its echelon form use gauss elimination method on the matrix and convert it into an upper triangular matrix, which will be in echelon form. Then count the number of non-zero rows in the upper triangular matrix to get the rank of the matrix.

(i) Elementary transformations do not alter the rank of a matrix.
(j) Only null matrix can have a rank of zero. All other matrices have rank of atleast one.
(k) Similar matrices have the same rank.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.38 Given Matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is

(a) 4
(b) 3
(c) 2
(d) 1

[CE, GATE-2003, 1 mark]

Solution: (c)

Consider first 3 × 3 minors, since maximum possible rank is 3

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and
$$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Since all 3 × 3 minors are zero, now try 2 × 2 minors.

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

So, rank = 2

Q.39 The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ____.

[CE, GATE-2014 : 2 Marks, Set-2]

Solution :

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1 + R_2$

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14-2(6)+(-2) & -14-2(0)+(14) & 0-2(4)+8 & -10-2(4)+(18) \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determinant of matrix $\begin{bmatrix} 6 & 0 \\ -2 & 14 \end{bmatrix}$ is not zero.

$\therefore$ Rank is 2.

Q.40 Let $A = [a_{ij}], 1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i.j$. The rank of A is

(a) 0
(b) 1
(c) $n - 1$
(d) n

[CE, GATE-2015 : 1 Mark, Set-II]

Solution: (b)

Rank of A = 1

Because each row will be scalar multiple of first row. So we will get only one non-zero row in row Echeleaon form of A.

Alternative:

Rank of A = 1

Because all the minors of order greater than 1 will be zero.

Q.41 $X = [x_1, x_2, \ldots x_n]^t$ is an n-tuple nonzero vector. The $n \times n$ matrix $V = XX^t$

(a) has rank zero
(b) has rank 1
(c) is orthogonal
(d) has rank n

[EE, GATE-2007, 1 mark]

Solution: (b)

If $X = (x_1, x_2, \ldots x_n)^t$

Rank X = 1, since it is non-zero n-tuple.

Rank X^t = Rank X = 1

Now Rank $(X^t) \le \min(\text{Rank } X, \text{Rank } X^t)$

$\Rightarrow$ Rank $(XX^t) \le \min(1, 1)$

$\Rightarrow$ Rank $(XX^t) \le 1$.

So XX^t has a rank of a either 0 or 1.

But since both X and X^t are non-zero vectors, so neither of their ranks can be zero.

So XX^t has a rank 1.

Q.42 Two matrices A and B are given below:

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix};\ B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$$

If the rank of matrix A is N, then the rank of matrix B is

(a) $\frac{N}{2}$ (b) N – 1
(c) N (d) 2 N

[EE, GATE-2014 : 1 Mark, Set-3]

Solution : (c)

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix} = AA^t$$

There are three cases for the rank of A

Case I: rank (A) = 0

⇒ A is null. So B = AA^t also has to be null and hence rank (B) is also equal to 0. Therefore in this case rank (A) = rank (B).

Case II: rank (A) = 1

⇒ A cannot be null. So B also cannot be null, since B = AA^t

and $|B| = |AA^t| = |A| \cdot |A^t| = |A|^2$

So rank (B) ≠ 0. Now since rank (A) ≠ 2 in this case, $|A| = 0$, which means that $|B| = |A|^2 = 0$

So rank (B) is also ≠ 2. Now since rank (B) ≠ 0 and ≠ 2, therefore rank (B) must be equal to 1.

Therefore in this case also rank (A) = rank (B).

Case III: rank (A) = 2

So A has to be non-singular, i.e. $|A| \neq 0$. Therefore, $|B| = |A|^2$ is also ≠ 0. So rank (B) = 2.

Therefore in this case also rank (A) = rank (B).

Therefore, in all three cases rank (A) = rank (B). So rank of A is N, then the rank of matrix B is also N.

Q.43 The dimension of the null space of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

[IN, GATE-2013 : 1 mark]

Solution: (b)

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{3\times 3}$$

order of matrix = 3

Rank = 2

∴ dimension of null space of A = 3 – 2 = 1.

Q.44 The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is

(a) 0
(b) 1
(c) 2
(d) 3

[EC, GATE-2006, 1 mark]

Solution: (c)

Perform, Gauss elimination

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is in row Echelon form

So its rank is the number of non-zero rows in this form.

i.e., rank = 2

Q.45 Let A be a 4×3 real matrix with rank 2. Which one of the following statement is TRUE?

(a) Rank of A^TA is less than 2.
(b) Rank of A^TA is equal to 2.
(c) Rank of A^TA is greater than 2.
(d) Rank of $A^T A$ can be any number between 1 and 3.

[EE, 2016 : 2 Marks, Set-1]

Solution: (b)

Result, $\text{Rank}(A^T A) = \text{Rank}(A)$

1.5.1 Elementary Matrices

A matrix obtained from a unit matrix by a single elementry transformation is called an elementry matrix.

1.5.2 Results

1. Elementary transformations do not change the rank of a matrix.
2. Two matrices are equivalent if one can be obtained from another by elementary row or column transformations. Equivalent matrices have same rank, since elementary transformations do not change the rank.
3. The rank of a product of two matrices cannot exceed the rank of either matrix. i.e. $r(AB) \le r(A)$ and $r(AB) \le r(B)$.
4. Rank of sum of two matrices cannot exceed the sum of their ranks. $r(A+B) \le r(A) + r(B)$.
5. If A, B are two n-rowed square matrices then $\text{Rank}(AB) \ge (\text{Rank } A) + (\text{Rank } B) - n$.

1.6 SUB-SPACES : BASIS AND DIMENSION

1.6.1 Introduction

A matrix can be thought of as an array of its rows as also an array of its columns. Further a row as well as a column is an ordered set of numbers. This view of matrix as an array of ordered sets of rows and columns is very useful in dealing with various linear problems. This chapter will be devoted to consideration of such ordered sets of numbers.

1.6.2 Ordered sets of numbers

Apart from the above context, we have also often to deal with ordered sets of numbers in other connections. Thus point in a plane and in space can respectively be represented by ordered pairs and triads of numbers; the numbers being the Cartesian coordinates of the points. Again, some of the physical concepts such as velocity, acceleration, force, etc., can also be represented as ordered triads of numbers which are the resolved parts along three coordinate axes.

In addition to considering ordered sets of numbers, we shall also consider two important compositions of Multiplication with Scalars and Addition of the same. Thus, if (a_1, a_2, a_3) be an ordered triad representing a force P, then the ordered triad $(ka_1, ka_2\ ka_3)$ represents another force whose magnitude is |k| times that of P and which acts along the line of action of P and whose sense is the same or opposite to that of P according as k is positive or negative. Again, if (a_1, a_2, a_3) and (b_1, b_2, b_3) be two ordered triads representing two forces acting at a point, then the triads $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$ represents the resultant of the two forces, for the sum of the resolved parts of forces along any line equals the resolved part of the resultant of the forces along the same line.

These considerations illustrate the important and useful principle that the same piece of Mathematics can be interpreted in several ways in relation to different applications.

We shall now proceed to consider the generalization to the arbitrary ordered sets of numbers to be called Vectors, and also the operations on the same.

Usually an entity having both magnitude and direction is called a Vector. We know, however, that this physical concept of a vector leads to a representation of the same by means of a triad of numbers.

1.6.3 Vector

Def. An ordered n-tuple of numbers is called an n-vector. The n numbers which are called components of the n-vector may be written in a horizontal or in a vertical line, and thus a vector will appear either as a row or a column matrix. A vector whose components belong to a field F is said to be over F. A vector over the field of real numbers is called a Real vector and that over the complex field is called a complex vector.

The n-vector space. The set of all n-vectors over a field F, to be denoted by $V_n(F)$, is called the n-vector space over F. The elements of the field F will be known as scalars relatively to the vector space.

1.6.4 Compositions in $V_n(F)$

The multiple $k\xi$ of an n-vector ξ by a scalar k is the n-vector whose components are the products by k of the components of ξ.

The sum $\xi_1 + \xi_2$ of two n-vectors ξ_1, ξ_2 is the n-vector whose components are the sums of the corresponding components of ξ_1 and ξ_2.

As these two compositions on n-vectors are only special cases of the same on matrices, their laws are the same as those in the case of matrices for those compositions.

It is obvious that if ξ_1, ξ_2 are two n-vectors over F, and k_1, k_2 are two members of F.

Example:

Given,

$$\xi_1 = [2\ 3\ 0\ 1],$$
$$\xi_2 = [3\ 1\ -1\ 2],$$
$$\xi_3 = [9\ 10\ -1\ 5]$$

compute the vectors

1. $2\xi_1 + 4\xi_2$
2. $3\xi_1 + 4\xi_2 + 5\xi_3$
3. $3\xi_1 + \xi_2 - \xi_3$

1.6.5 Linearly dependent and Linearly Independent Sets of Vectors

1.6.5.1 Linearly Dependent Sets of Vectors

Def. A set $\{\xi_1, \xi_2, ...\xi_r\}$ of r vectors is said to be a linearly dependent set, if there exist r scalars k_1, k_2, ..., k_r.

Not all zero, such that $k_1\xi_1 + k_2\xi_2 + ... + k_r\xi_r = 0$ where, zero, denotes the n-vector with components all zero.

1.6.5.2 Linearly Independent Sets of Vectors

Def. A set $\{\xi_1, \xi_2, ...\xi_r\}$ of r vectors is said to be a linearly independent set, if the set, is not linearly dependent, i.e. if $k_1\xi_1 + k_2\xi_2 + ... + k_r\xi_r = 0$.

$\Rightarrow \quad k_1 = 0, k_2 = 0, ..., k_r = 0$

1.6.5.3 A vector as a Linear Combination of a Set of Vectors

Def. A vector ξ which can be expressed in the form $\{\xi = k_1\xi_1 + ... + k_r\xi_r\}$ is said to be a linear combination of the set $\{\xi_1, \xi_2, ..., \xi_r\}$ of vectors.

Example: Given a linearly dependent set of vectors, show that at least one member of the set is a linear combination of the remaining members of the set.

Example:

1. Show that the vectors [1 2 3], [2 – 2 0] from a linearly independent set.
2. Show that the vectors [2 3 –1 –1], [1 –1 –2 –4], [3 1 3 –2], [6 5 0 – 7] form a linearly dependent set. Also express one of these as a linear combination of the others.
3. Show that the set consisting only of the zero vector, O, is linearly dependent.

Solution:

1. Consider the relation

$$k_1[1\ 2\ 3] + k_2[2\ -2\ 0] = \text{zero}$$

This relation is equivalent to the ordinary system of liner equations

$$k_1 + 2k_2 = 0,\ 2k_1 - 2k_2 = 0,\ 3k_1 = 0$$

As $k_1 = 0, k_2 = 0$ are the only values of k_1, k_2 which satisfy these three equations, we see that the given set is linearly independent.

2. The single relation

$$k_1\xi_1 + k_2\xi_2 + k_3\xi_3 + k_4\xi_4 = 0 \quad ...(i)$$

where $\xi_1, \xi_2, \xi_3, \xi_4$ are the four given vectors in the given order, is equivalent to

$$2k_1 + k_2 + 3k_3 + 6k_4 = 0,\ 3k_1 - k_2 + k_3 + 3k_4 = 0$$
$$-k_1 - 2k_2 + 3k_3 + 0k_4 = 0,\ -k_1 - 4k_2 - 2k_3 - 7k_4 = 0$$

As this system of 4 linear equations is satisfied by the values

$k_1 = 1, k_2 = 1, k_3 = 1, k_4 = -1$ (after solving above system) ... (ii)

which are not all zero, the given vectors form a linearly dependent set.

Also we have the relations,

$\xi_1 + \xi_2 + \xi_3 - \xi_4 = 0$ (obtained by substituting (ii) in (i))

by means of which any one of the four given vectors can be expressed as a linear combination of the remaining three others.

3. Let X = (0, 0, 0,.....,0) be an n-vector whose components are all zero. Then the relatk n kX = 0 is true for some non-zero value of the number k, For example 2x = 0 and 2 ≠ 0.

Hence the vector 0 is linearly dependent.

1.6.6 Some properties of linearly Independent and Dependent Sets of Vectors

In the following, it is understood that the vectors belong to a given vector space $V_n(F)$.

1. If η is a linear combination of the set $\{\xi_1, \ldots, \xi_r\}$, then the set $\{\eta, \xi_1, \xi_2, \ldots, \xi_r\}$ is linearly dependent we have

$$\eta = k_1\xi_1 + k_2\xi_2 + \ldots + k_r\xi_r$$

$$\Rightarrow \quad \eta - k_1\xi_1 - k_2\xi_2 - \ldots - k_r\xi_r = 0$$

As at least one of the coefficients, viz., that of η, in this latter relation is not zero, we establish the linear dependence of the set

$$\{\eta, \xi_1, \ldots, \xi_r\}$$

2. Also, If $\{\xi_1, \ldots, \xi_r\}$ is a linearly independent and $\{\xi_1, \ldots, \xi_r, \eta\}$ is a linearly dependent set, then η is a linear combination of the set $\{\xi_1, \ldots, \xi_r\}$.
3. Every super-set of a linearly dependent set is linearly dependent.
4. It may also be easily shown that every sub-set of a linearly independent set is linearly independent.

1.6.7 Subspaces of an N-vector space V_n

Definition: Any non-empty set S, of vectors of $V_n(F)$ is called a subspace of $V_n(F)$, if when

1. ξ_1, ξ_2 are any two members of S, then $\xi_1 + \xi_2$ is also a member of S; and
2. ξ is a member of S, and k is a scalar, then $k\xi$ is also a member of S.

Briefly, we may say that a set S of vectors of $V_n(F)$ is a subspace of $V_n(F)$ it it closed w.r.t. the compositions of "addition" and "multiplication with scalars".

Every subspace of V_n contains the zero vector; being the product of any vector with the scalar zero.

Example: $\xi = [a, b, c]$ is a non-zero vector of V_3. Show that the set of vectors $k\xi$ is a subspace of V_3; k being variable.

1.6.7.1 Construction of Subspaces

Theorem 1: The set S, of all linear combinations of a given set of r fixed vectors of V_n is a subspace of V_n.

Def. 1 A subspace Spanned by a Set of Vectors. A subspace which arises as a set of all linear combinations of any given set of vectors, is said to be spanned by the given set of vectors.

Def. 2. Basis of a Subspace. A set of vectors is said to be a basis of a subspace, if

1. the subspace is spanned by the set, and
2. the set is linearly independent.

It is important to notice that the set of vectors

$e_1 = [1\ 0\ 0 \ldots 0]$, $e_2 = [0\ 1\ 0 \ldots 0]$, ..., $e_n = [0\ 0 \ldots 0\ 1]$

is a basis of the vector space V_n, for, if

$$k_1e_1 + k_2e_2 + \ldots + k_ne_n = 0$$

then, $k_1 = 0, \ldots, k_n = 0$ so that the set is linearly independent and any vector

$$\xi = [a_1, a_2, \ldots, a_n]$$

of V_n is expressible as

$$\xi = a_1e_1 + a_2e_2 + \ldots + a_ne_n$$

Theorem 2: A basis of a subspace, S, can always be selected from a set of vectors which span S.

Let $\{\xi_1, \ldots, \xi_r\}$

be a set of vectors which span a subspace S.

If this set is linearly independent, then it is already a basis. In case it is linearly dependent, then some member of the set is a linear combination of the preceding members. Deleting this member, we obtain another set which also spans S.

Continuing in this manner, we shall ultimately, in a finite number of steps, arrive at a basis of S.

Note: It has yet to be shown that every subspace, S, of V_n possesses a basis and that the number of vectors in every basis of S, is the same.

Example: Show that the following two sets of vectors span the same subspace of $V_3(F)$.

1. $\{[2\ \ -1\ \ 4], [0\ \ 1\ \ 2]\}; \{[6\ \ -1\ \ 18], [4\ \ 0\ \ 12]\}$

   Same question for the sets of vectors for the same subspace of $V_4(F)$.

2. $\{[2\ \ 3\ \ 4], [1\ \ 2\ \ 3\ \ 4]\}; \{[2\ \ 5\ \ 8\ \ 11], [3\ \ 5\ \ 7\ \ 9]\}$

Example: Show that the following set of vectors constitute a basis of V_3:

$$\{[2\ \ 3\ \ 4], [0\ \ 1\ \ 2], [-1\ \ 1\ \ -1]\}$$

Example: Determine a basis of the subspace spanned by the vectors:

$$\{[2\ \ -3\ \ 1], [3\ \ 0\ \ 1], [0\ \ 2\ \ 1], [1\ \ 1\ \ 1]\}$$

Invariant Character of the Number of Vectors in a Basis:

Result 1: The number of members in any one basis of a subspace is the same as in any other basis.

Result 2: Every basis of V_n possesses n members, for, as seen before, V_n possesses one basis of n members.

Theorem 3: Every linear independent set of vectors $\{\xi_1, \xi_2, ..., \xi_r\}$ can be extended so as to constitute a basis of V_n.

Result 3: Every set of (n + 1) Vectors of V_n is Linearly Dependent: Either the set is linearly dependent or linearly independent. In the case of linear independence, the set can be extended so as to constitute a basis of V_n (by theorem 3 above) and the basis thus obtained will contain at least (n +1) members, but this is not possible (since, every basis of V_n possesses exactly n members). Thus the set must be linearly dependent.

Result 4: Existence of a basis: Every subspace, S, of V_n has a basis.

Note: The number of vectors in any basis of a subspace is called the dimension of the subspace. In particular, we see that the dimension of V_n is n.

1.6.8 Row and column spaces of a matrix. Row and column ranks of a Matrix

Let A, be any m × n matrix over a field F.

Each of the m rows of A, consisting of n elements, is an n-vector and is as such a member of $V_n(F)$. The space spanned by the m rows which is a subspace of V_n is called the Row space of the m × n matrix A.

Again each of the n columns consisting of m elements is an m-vector and is a member of $V_m(F)$. The space spanned by the n columns which is a subspace of V_m is called the Column space of the m × n matrix A.

The dimensions of these row and column spaces of matrix are respectively called the Row rank and the Column rank of the matrix.

Theorem 1: Pre-multiplication by a non-singular matrix does not alter the rank of a matrix.

In a similar manner, we may prove that post-multiplication with a non-singular matrix does not alter the column rank of a matrix.

1.6.8.1 Equality of row rank, column rank and rank

Theorem 2: The row rank of a matrix is the same as its rank.

Theorem 3: The column rank of a matrix is the same as its rank.

Corollary 1: The rank of a matrix is equal to the maximum number of its linearly independent rows and also to the maximum number of its linearly independent columns. Thus a matrix of rank r, has a set of r linearly independent rows (columns), such that each of the other rows (columns), is a linear combination of the same.

Corollary 2: The rows and columns of an n-rowed non-singular square matrix form linearly independent sets and are as such bases of V_n.

1.6.8.2 Connection between Rank and Span

A set of n vectors $X_1, X_2, X_3 \ldots X_n$ spans R^n iff they are linearly independent which can be checked by constructing a matrix with $X_1, X_2, X_3 \ldots X_n$ as its rows (or columns) and checking that the rank of such a matrix is indeed n. If however the rank is less than n, say m, then the vectors span only a subspace of R^n.

Example: Cheek of the vectors $[1\ \ 2\ \ -1], [2\ \ 3\ \ 0], [-1\ \ 2\ \ 5]$ span R^3.

Solution:

Step 1: Construct a matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

Step 2: Find its rank

Since
$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 2 & 5 \end{vmatrix} = 1(15-0) - 2(10-0) - 1(4+3)$$
$$= 15 - 20 - 7 = -12$$
$$\neq 0$$

So, rank $= 3$

$\therefore$ The vectors are linearly independent and hence span R^3.

Example: Cheek if the vectors $[1\ \ 2\ \ 3], [4\ \ 5\ \ 6]$ and $[7\ \ 8\ \ 9]$ span R^3.

Solution:

Since,
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

has a
$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$
$$= 1(45-48) - 2(36-42) + 3(32-45)$$
$$= 0$$

So its rank $\neq 3$

Since,
$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$$

$\therefore$ Rank $= 2$

So the vectors $[1\ \ 2\ \ 3], [4\ \ 5\ \ 6]$ and $[7\ \ 8\ \ 9]$ span a subspace of R^3 but do not span R^3.

1.6.9 Orthogonality of Vectors

1. Two vectors X_1 and X_2 are orthogonal iff each is non zero and the dot product $X_1' X_2 = 0$.

   Example: The vectors $[a \ b \ c]$ and $[d \ e \ f]$ are orthogonal iff

   $$[a \ b \ c]^t \times [d \ e \ f] = 0$$

   i.e.
   $$ad + be + cf = 0$$

   Example: The vectors $[1 \ 2]$ and $[-2 \ 1]$ are orthogonal since

   $$[1 \ 2]^t \times [-2 \ 1] = \begin{bmatrix}1\\2\end{bmatrix} \times [-2 \ 1]$$
   $$= (1 \times -2) + (2 \times 1)$$
   $$= 0$$

   Example: The vectors $[1 \ 2 \ 3]$ and $[-1 \ 2 \ 5]$ are not orthogonal since

   $$(1 \times -1) + (2 \times 2) + (3 \times 5) = 18 \neq 0$$

2. Three vectors X_1, X_2 and X_3 are orthogonal iff each is non zero and they are pairwise orthogonal.

   i.e. $X_1' X_2 = 0$

   and $X_1' X_3 = 0$

   and $X_2' X_3 = 0$

   Example: The vectors $[1 \ 0 \ 0]$, $[0 \ 1 \ 0]$ and $[0 \ 0 \ 1]$ are orthogonal since

   $$[1 \ 0 \ 0]^t [0 \ 1 \ 0] = [0 \ 0 \ 0]$$

   and $$[0 \ 1 \ 0]^t [0 \ 0 \ 1] = [0 \ 0 \ 0]$$

   and $$[1 \ 0 \ 0]^t [0 \ 0 \ 1] = [0 \ 0 \ 0]$$

3. If n vectors $X_1, X_2, X_3, \ldots X_n$ each of which is in R^n, are orthogonal, then they are surely linearly independent and hence span R^n and therefore form a basis for R^n.

   Example: The vectors $[1 \ 0 \ 0]$, $[0 \ 1 \ 0]$ and $[0 \ 0 \ 1]$ are orthogonal and hence are linearly independent and hence span R^3. They form a basis for R^3.

   The vectors $[0 \ -2]$, $[-2 \ 0]$ are orthogonal and hence are linearly independent and span R^2 and form a basis of R^2.

4. The set of n vectors $X_1, X_2, X_3, \ldots X_n$ are called orthonormal if they are

   (a) orthogonal and

   (b) if each vector has unit length.

   The two conditions together can be written as

   $$X_i' . X_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

   A set of orthogonal vectors X can be converted to a set of orthonormal vectors by divising each vector in the orthogonal set by its length (Euclidean norm $\|X\|$).

   Example: The set [1, 2, 1], [2, 1, –4] and [3, –2, 1] is an orthogonal basis of vectors for R^3, since these are pairwise orthogonal and hence are linearly independent and hence span R^3.

   To convert this set to an orthonormal basis of R^3, we need to divide each vector by its length

   $$\|u_1\| = \sqrt{1+4+1} = \sqrt{6}$$
   $$\|u_2\| = \sqrt{4+1+16} = \sqrt{21}$$

$$||u_3|| = \sqrt{9+4+1} = \sqrt{14}$$

So an orthonormal basis of R^3 is $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-4}{\sqrt{21}}\right)$ and $\left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.46 Choose the **CORRECT** set of functions, which are linearly dependent.

(a) sin x, $\sin^2$ x and $\cos^2$ x
(b) cos x, sin x and tan x
(c) cos 2 x, $\sin^2$ x and $\cos^2$ x
(d) cos 2x, sin x and cos x

[ME, GATE-2013, 1 Mark]

Solution: (c)

Since, $\cos 2x = \cos^2 x - \sin^2 x$, therefore cos 2 x is a linear combination of $\sin^2$ x and $\cos^2$ x and hence these are linearly dependent.

Q.47 $P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T, Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$ and $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$ are three vectors.

An orthogonal set of vectors having a span that contains P, Q, R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Solution: (a)

We are looking for orthogonal vectors having a span that contain P, Q and R.

Take choice (a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

Firstly these are orthogonal, as can be seen by taking their dot product

$= -6 \times 4 + -3 \times -2 + 6 \times 3 = 0$

The space spanned by these two vectors is

$$k_1 \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \quad \text{... (i)}$$

The span of $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ contains P, Q and R. We can show this by successively setting equation (i) to P, Q and R one by one and solving for k_1 and k_2 uniquely.

Notice also that choices (b), (c) and (d) are wrong since none of them are orthogonal as can be seen by taking pairwise dot products.

Q.48 The following vector is linearly dependent upon the solution to the previous problem

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Solution: (b)

The vector $[-2 \quad -17 \quad 30]^t$ is linearly dependent upon the solution obtained in previous question namely $[-6 \quad -3 \quad 6]^t$ and $[4 \quad -2 \quad 3]^t$.

This can be easily cheeked by finding determinant of $\begin{bmatrix} -6 & -3 & 6 \\ 4 & -2 & 3 \\ -2 & -17 & 30 \end{bmatrix}$.

$$\begin{vmatrix} -6 & -3 & 6 \\ 4 & -2 & 3 \\ -2 & -17 & 30 \end{vmatrix} = -6(-60+51)+3(120+6)+6(-68-4)=0$$

Hence, it is linearly dependent.

Q.49 If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct?

(a) Q will have four linearly independent rows and four linearly independent columns
(b) Q will have four linearly independent rows and five linearly independent columns
(c) QQ^T will be invertible
(d) Q^TQ will be invertible

[EE, GATE-2008, 1 mark]

Solution: (a)

If rank of (5×6) matrix is 4, then surely it must have exactly 4 linearly independent rows as well as 4 linearly independent columns, since rank = row rank = column rank.

Q.50 It is given that $X_1, X_2, \ldots X_M$ are M non-zero, orthogonal vectors. The dimension of the vector space spanned by the 2M vectors $X_1, X_2, \ldots X_M, -X_1, -X_2, \ldots -X_M$ is

(a) 2M (b) M + 1
(c) M (d) dependent on the choice of $X_1, X_2, \ldots X_M$

[EC, GATE-2007, 2 marks]

Solution: (c)

Since $(X_1, X_2, \ldots X_M)$ are orthogonal, they span a vector space of dimension M.
Since $(-X_1, -X_2, \ldots X_M)$ are linearly dependent on $X_1, X_2, \ldots X_M$, the set $(X_1, X_2, X_3, \ldots X_M, -X_1, -X_2, \ldots -X_M)$ will also span a vector space of dimension M only.

Q.51 Consider the set of (column) vectors defined by $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0$, where $x^T = [x_1, x_2, x_3]^T\}$. Which of the following is TRUE?

(a) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a basis for the subspace X.
(b) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X
(c) X is not a subspace for R^3
(d) None of the above

[CS, GATE-2007, 2 marks]

Solution: (a)

To be basis for subspace X, two conditions are to be satisfied

1. The vectors have to be linearly independent.
2. They must span X.

Here,
$$X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$$
$$x^T = [x_1, x_2, x_3]^T$$

Step 1: Now, $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set because one cannot be obtained from another by scalar multiplication. The fact that it is independent can also be established by seeing that rank of $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is 2.

Step 2: Next, we need to check if the set spans X.

Here, $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

The general infinite solution of $X = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$

Choosing k_1, k_2 as $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ and $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$, we get 2 linearly independent solutions, for X,

$$X = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} \text{ or } \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix}$$

Now since both of these can be generated by linear combinations of $[1, -1, 0]^T$ and $[1, 0, -1]^T$, the set spans X. Since we have shown that the set is not only linearly independent but also spans X, therefore by definition it is a basis for the subspace X.

Q.52 Let $P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Consider the set S of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $a^2 + b^2 = 1$ where $\begin{pmatrix} a \\ b \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$. Then S is

(a) a circle of radius $\sqrt{10}$

(b) a circle of radius $\frac{1}{\sqrt{10}}$

(c) an ellipse with major axis along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) an ellipse with minor axis along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

[EE, 2016 : 2 Marks, Set-2]

Solution: (c)

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$
$$3x + y = a$$
$$x + 3y = b$$
$$a^2 + b^2 = 1$$
$$\Rightarrow \quad 10x^2 + 10y^2 + 12xy = 1$$

Ellipse with major axis along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Q.53 If the vectors $e_1 = (1, 0, 2)$, $e_2 = (0, 1, 0)$ and $e_3 = (-2, 0, 1)$ form an orthogonal basis of the three-dimensional real space R^3, then the vector $u = (4, 3, -3) \in R^3$ can be expressed as

(a) $u = -\frac{2}{5}e_1 - 3e_2 - \frac{11}{5}e_3$

(b) $u = -\frac{2}{5}e_1 - 3e_2 + \frac{11}{5}e_3$

(c) $u = -\frac{2}{5}e_1 + 3e_2 + \frac{11}{5}e_3$

(d) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

Solution: (d)

$$\begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$a - 2c = 4$$
$$b = 3$$
$$2a + c = -3$$

from here

$$a = -\frac{2}{5}$$
$$b = 3$$
$$c = -\frac{11}{5}$$
$$u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$$

1.7 SYSTEM OF LINEAR EQUATIONS

1.7.1 Homogenous Linear Equations

Suppose,

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = 0 \end{matrix}\right\} \quad \dots \text{(i)}$$

is a system of m homogenous equations in n unknowns $x_1, x_2, \dots x_n$.

Let

$$A = \begin{bmatrix} a_{11} \; a_{12} \dots\dots a_{1n} \\ a_{21} \; a_{22} \dots\dots a_{2n} \\ \dots\dots\dots\dots \\ a_{m1} \; a_{m2} \dots\dots a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

where A, X, O are $m \times n, n \times 1, m \times 1$ matrices respectively. Then obviously we can write the system of equations in the form of a single matrix equation $AX = 0$ ... (ii)

The matrix A is called coefficient matrix of the system of equation (i).

The set $S = \{x_1 = 0, x_2 = 0, \ldots x_n = 0\}$ i.e., $X = 0$ is always a solution of equation (i).

But in general there may be infinite number of solutions to equation (ii).

Again suppose X_1 and X_2 are two solutions of (ii). Then their linear combination, $R_1X_1 + R_2X_2$ when R_1 and R_2 are any arbitrary numbers, is also solution of (ii).

1.7.1.1 Important Results

The number of linearly independent infinite solutions of m homogenous linear equations in n variables, $AX = 0$, is $(n - r)$, where r is rank of matrix A.

$n - r$ is also the number of parameters in the infinite solution.

1.7.1.2 Some important results regarding nature of solutions of equation A X = O

Suppose there are m equations in n unknowns. Then the coefficient matrix A will be of the type $m \times n$. Let r be rank of matrix A. Obviously r cannot greater than n. Therefore we have either $r = n$ or $r < n$.

Case 1: ***Inconsistency***: This is not possible in a homogeneous system since such a system is always consistent (since the trivial solution $X = [0, 0, 0...]^t$ always exists for a homogeneous system).

Case 2: ***Consistent Unique Solution***: If $r = n$; the equation $AX = O$ will have only the trivial unique solution $X = [0, 0, 0...]^t$.

Note: That $r = n \Rightarrow |A| \neq 0$ i.e. A is non-singular.

Case 3: ***Consistent Infinite Solution:*** If $r < n$ we shall have $n - r$ linearly independent non-trivial infinite solutions. Any linear combination of these $(n - r)$ solutions will also be a solution of $AX = O$. Thus in this case, the equation $AX = O$ will have infinite solutions.

Note: That $r < n \Rightarrow |A| = 0$ i.e. A is a singular matrix.

1.7.2 System of Linear Non-Homogeneous Equations

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \ldots (i)$$

be a system of m non-homogenous equations in n unknown, $x_1, x_2 \ldots x_n$.

If we write
$$A = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots a_{1n} \\ a_{21} & \ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{m \times 1}$$

where A, X, B are $m \times n$, $n \times 1$, and $m \times 1$ matrices respectively. The above equations can be written in the form of a single matrix equation A X = B.

"Any set of values of x_1, x_2 .... x_n which simultaneously satisfy all these equation is called a solutions of the system. When the system of equations has one or more solutions, the equation are said to be consistent otherwise they are said to be inconsistent".

$$\text{The matrix } [A \ \ B] = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} & b_1 \\ a_{21} & a_{22} \ldots a_{2n} & b_2 \\ \ldots & \ldots\ldots & \\ a_{m1} & a_{m2} \ldots a_{mn} & b_m \end{bmatrix}$$

is called augmented matrix of the given system of equations.

Condition for Consistency: The system of equations AX = B is consistent i.e., possess a solution iff the coefficient matrix A and the augmented matrix [A B] are of the same rank. i.e. r(A) = r(A, B).

Case 1: ***Inconsistency:*** If $r(A) \neq r(A|B)$ the system A x = B, has no solution. We say that such a system is inconsistent.

Cases 2 and 3: ***Consistent systems:*** Now, when $r(A) = r(A|B) = r$., The system is consistent and has solution.

We say, that the rank of the system is r. Now two cases arise.

Case 2: ***Consistent Unique Solution:*** If $r(A) = r(A|B) = r = n$ (where n is the number of unknown variables of the system), then the system is not only consistent but also has a unique solution.

Case 3: ***Consistent Infinite solution:*** If $r(A) = r(A|B) = r < n$, then the system is consistent, but has infinite number of solutions.

In summary we can say the following:

1. If $r(A) \neq r(A|B)$ (Inconsistent and hence, no solution)
2. If $r(A) = r(A|B) = r = n$ (consistent and unique solution)
3. If $r(A) = r(A|B) = r < n$ (consistent and infinite solution)

The rank of a system of equations as well as its solution (if it exists) can be obtained by a procedure called Gauss - Elimination method, which reduces the matrix A to its Echelon form and then by counting the number of non-zero rows in that matrix we get the rank of A.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.54 Consider a non-homogeneous system of linear equations representing mathematically an over-determined system. Such a system will be

(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

[CE, GATE-2005, 1 mark]

Solution: (a), (b) and (d) all possible.

In an over determined system having more equations than variables, all three possibilities still exist (a) consistent unique (b) consistent infinite and (d) in consistent with no situation.

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Q.55 Solution for the system defined by the set of equations $4y + 3z = 8$, $2x - z = 2$ and $3x + 2y = 5$ is

(a) $x = 0; y = 1; z = 4/3$
(b) $x = 0; y = 1/2; z = 2$
(c) $x = 1; y = 1/2; z = 2$
(d) non-existent

[CE, GATE-2006, 1 mark]

Solution: (d)

The augmented matrix for given system is

$$\left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right] \xrightarrow{\text{Exchange 1st and 2nd row}} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

then by Gauss elimination procedure

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right] \xrightarrow{R_3 - \frac{3}{2}R_1} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 2 & 3/2 & 2 \end{array}\right] \xrightarrow{R_3 - \frac{2}{4}R_2} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 8 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

For last row we see $0 = -2$ which is inconsistent. Also notice that $r(A) = 2$, while $r(A \mid B) = 3$, ($r(A) \neq r(A \mid B)$ means inconsistent).

$\therefore$ Solution is non-existent for above system.

Q.56 For what values of α and β, the following simultaneous equations have an infinite number of solutions?

$x + y + z = 5$; $x + 3y + 3z = 9$; $x + 2y + \alpha z = \beta$

(a) 2, 7
(b) 3, 8
(c) 8, 3
(d) 7, 2

[CE, GATE-2007, 2 marks]

Solution: (a)

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right]$$

Using Gauss-elimination method we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha - 1 & \beta - 5 \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha - 2 & \beta - 7 \end{array}\right]$$

Now, for infinite solution last row must be completely zero

i.e. $\alpha - 2$ and $\beta - 7 = 0$

$\Rightarrow$ $\alpha = 2$ and $\beta = 7$

Q.57 The following simultaneous equations

$x + y + z = 3$
$x + 2y + 3z = 4$
$x + 4y + kz = 6$

will NOT have a unique solution for k equal to

(a) 0
(b) 5
(c) 6
(d) 7

[CE, GATE-2008, 2 marks]

Solution: (d)

The augmented matrix for given system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right]$.

Using Gauss elimination we reduce this to an upper triangular matrix to investigate its rank.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & k-1 & 3 \end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-7 & 0 \end{array}\right]$$

Now if $k \neq 7$

$$\text{rank}(A) = \text{rank}(A \mid B) = 3$$

∴ unique solution

If $k = 7$, rank (A) = rank (A | B) = 2

which is less than number of variables

∴ when $k = 7$, unique solution is not possible and only infinite solution is possible.

Q.58 The eigen values of matrix $\begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$ are

(a) –2.42 and 6.86
(b) 3.48 and 13.53
(c) 4.70 and 6.86
(d) 6.86 and 9.50

[CE, GATE-2012, 2 marks]

Solution: (b)

We need eigen values of $A = \begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$

The characteristic equation is

$$\begin{vmatrix} 9-\lambda & 5 \\ 5 & 8-\lambda \end{vmatrix} = 0$$

$$(9-\lambda)(8-\lambda) - 25 = 0$$

$$\Rightarrow \quad \lambda^2 - 17\lambda + 47 = 0$$

So eigen values are,

$$\lambda = 3.48, 13.53$$

Q.59 For what value of p the following set of equations will have no solution?

$$2x + 3y = 5$$
$$3x + py = 10$$

[CE, GATE-2015 : 1 Mark, Set-I]

Solution:

Given system of equations has no solution if the lines are parallel i.e., their slopes are equal

$$\frac{2}{3} = \frac{3}{p}$$

$$\Rightarrow \quad p = 4.5$$

Q.60 Consider the system of simultaneous equations

$x + 2y + z = 6$; $2x + y + 2z = 6$; $x + y + z = 5$

This system has

(a) unique solution
(b) infinite number of solutions
(c) no solution
(d) exactly two solutions

[ME, GATE-2003, 2 marks]

Solution: (c)

Given equation are

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

Given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$

By gauss elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{array}\right] \xrightarrow{R_3 - \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$r(A) = 2$$
$$r(A \mid B) = 3$$

Since the rank of coefficient matrix is 2 and rank of argument matrix is 3, which is not equal. Hence system has no solution i.e. system is inconsistent.

Q.61 A is a 3 × 4 real matrix and A x = b is an inconsistent system of equations. The highest possible rank of A is

(a) 1
(b) 2
(c) 3
(d) 4

[ME, GATE-2005, 1 mark]

Solution: (b)

$$r(A_{m \times n}) \le \min(m, n)$$

So, Highest possible rank = Least value of 3 and 4.

i.e. highest possible rank (based on size of A) = 3

However if the rank of A = 3 then rank of [A|B] also would be 3, which means the system would become consistent. But it is given that the system is inconsistent. So the maximum rank of A could only be 2.

Q.62 For what value of a, if any, will the following system of equations in x, y and z have a solution?

$2x + 3y = 4$; $x + y + z = 4$; $x + 2y - z = a$

(a) Any real number
(b) 0
(c) 1
(d) There is no such value

[ME, GATE-2008, 2 marks]

Solution: (b)

Augmented matrix is $\left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right]$.

Performing guess-elimination on this matrix, we get,

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right] \xrightarrow[R_3 - \frac{1}{2}R_1]{R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 0 & -1/2 & 1 & 2 \\ 0 & 1/2 & -1 & a-2 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 0 & -1/2 & 1 & 2 \\ 0 & 0 & 0 & a \end{array}\right]$$

If $a \neq 0$, $r(A) = 2$ and $r(A \mid B) = 3$, hence system will have no solutions.

If $a = 0$, $r(A) = r(A \mid B) = 2$, then the system will be consistent and will have solution (Infinite solution).

Q.63 Consider the following system of equations

$2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$

This system has

(a) a unique solution
(b) no solution
(c) infinite number of solutions
(d) five solutions

[ME, GATE-2011, 2 marks]

Solution: (c)

The Augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Performing gauss elimination on $[A \mid B]$ we get

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank (A) = Rank $(A \mid B) = 2 < 3$

So infinite number of solutions are obtained.

Q.64 $x + 2y + z = 4$

$2x + y + 2z = 5$

$x - y + z = 1$

The system of algebraic given below has

(a) A unique solution of $x = 1$, $y = 1$ and $z = 1$
(b) only the two solutions of $(x = 1, y = 1, z = 1)$ and $(x = 2, y = 1, z = 0)$
(c) infinite number of solutions
(d) no feasible solution

[ME, GATE-2012, 2 marks]

Solution: (c)

The given system is

$$x + 2y + z = 4$$
$$2x + y + 2z = 5$$
$$x - y + z = 1$$

Use Gauss elimination method as follows:

Augmented matrix is

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 5 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & -3 & 0 & -3 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Rank}(A) = 2$$
$$\text{Rank}(A/B) = 2$$

So $\text{Rank}(A) = \text{Rank}(A/B) = 2$

System is consistent

Now system rank $r = 2$

Number of variables $n = 3$

$$r < n$$

So we have infinite number of solutions.

Q.65 In the matrix equation $Px = q$, which of the following is a necessary condition for the existence of at least one solution for the unknown vector x

(a) Augmented matrix [Pq] must have the same rank as matrix P

(b) Vector q must have only non-zero elements

(c) Matrix P must be singular

(d) Matrix P must be square

[EE, GATE-2005, 1 mark]

Solution: (a)

Rank [Pq] = Rank [P] is necessary for existence of at least one solution to $Px = q$.

Q.66 For the set of equations

$x_1 + 2x_2 + x_3 + 4x_4 = 2$

$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$

the following statement is true:

(a) Only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$ exists

(b) There are no solution

(c) A unique non-trivial solution exists

(d) Multiple non-trivial solutions exist **[EE, GATE-2010, 2 marks]**

Solution: (d)

$$x_1 + 2x_2 + x_3 + 4x_4 = 2$$
$$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$$

The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{array}\right]$

Performing gauss-elimination on this we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{rank}(A) = \text{rank}(A \mid B) = 1$$

So, system is consistent.

Since, system's rank = 1 is less than the number of variables, only infinite (multiple) non-trivial solution exists.

Q.67 The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(a) no solution

(b) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) non-zero unique solution

(d) multiple solutions

[EE, GATE-2013, 1 Mark]

Solution: (d)

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 2x_2 = 0$$

$$x_1 - x_2 = 0$$

$$\Rightarrow \quad x_1 = x_2$$

i.e. x_1 and x_2 are are having infinite number of solutions.

$\Rightarrow$ Multiple solutions are these.

Q.68 Given a system of equations:

$$x + 2y + 2z = b_1$$

$$5x + y + 3z = b_2$$

Which of the following is true regarding its solution?

(a) The system has a unique solution for any given b_1 and b_2

(b) The system will have infinitely many solutions for any given b_1 and b_2

(c) Whether or not a solution exists depends on the given b_1 and b_2

(d) The system would have no solution for any values of b_1 and b_2

[EE, GATE-2014 : 1 Mark, Set-1]

Solution : (b)

The augmented matrix for this system is $\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -9 & -7 & b_2 - 5b_1 \end{array}\right]$

Now gauss elimination is completed. We can see that the Rank(A) = 2.

Rank(A|B) is also = 2 (does not depend on value of b_1 and b_2).

So Rank(A) = Rank(A|B) < Number of variables = 3

Therefore the system is consistent and as infinitely many solutions.

Q.69 We have a set of 3 linear equations in 3 unknowns. '$X \equiv Y$' means X and Y are equivalent statements and '$X \not\equiv Y$' means X and Y are not equivalent statements.

P : There is a unique solution.

Q : The equations are linearly independent.

R : All eigenvalues of the coefficient matrix are nonzero.
S : The determinant of the coefficient matrix is nonzero.
Which one of the following is TRUE?
(a) $P \equiv Q \equiv R \equiv S$
(b) $P \equiv R \not\equiv Q \equiv S$
(c) $P \equiv Q \not\equiv R \equiv S$
(d) $P \not\equiv Q \not\equiv R \not\equiv S$

[EE, GATE-2015 : 1 Mark, Set-2]

Solution: (a)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $|A| \neq 0$ then $AX = B$ can be written as $X = A^{-1}B$. It leads unique solutions.

If $|A| \neq 0$ then $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$ each λ_i is non-zero.

If $|A| \neq 0$ then all the row (column) vectors of A are linearly independent.

Q.70 The system of linear equations

$4x + 2y = 7$
$2x + y = 6$

has
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) exactly two distinct solutions

[EC, GATE-2008, 1 mark]

Solution: (b)

The system can be written in matrix from as

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

The Augmented matrix [A|B] is given by $\left[\begin{array}{cc|c} 4 & 2 & 7 \\ 2 & 1 & 6 \end{array}\right]$

Performing Gauss elimination on this [A|B] as follows:

$$\left[\begin{array}{cc|c} 4 & 2 & 7 \\ 2 & 1 & 6 \end{array}\right] \xrightarrow[= R_2 - \frac{1}{2}R_1]{R_2 - \frac{2}{4}R_1} \left[\begin{array}{cc|c} 4 & 2 & 7 \\ 0 & 0 & 5/2 \end{array}\right]$$

Now Rank [A|B] = 2 (The number of non-zero rows in [A|B]
Rank [A] = 1 (The number of non-zero rows in [A])
Since, Rank [A|B] $\neq$ Rank[A].
The system has no solution.

Q.71 The system of equations

$$x + y + z = 6 \quad ; \quad x + 4y + 6z = 20 \quad ; \quad x + 4y + \lambda z = \mu$$

has NO solution for values of λ and μ given by

(a) $\lambda = 6, \mu = 20$
(b) $\lambda = 6, \mu \neq 20$
(c) $\lambda \neq 6, \mu = 20$
(d) $\lambda \neq 6, \mu \neq 20$

Solution: (b) [EC, GATE-2011, 2 mark]

The augmented matrix for the system of equations is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 0 & 0 & \lambda-6 & \mu-20 \end{array}\right] \qquad [R_3 \to R_3 - R_2]$$

If $\lambda = 6$ and $\mu \neq 20$ then

$$\text{Rank}(A|B) = 3 \text{ and Rank}(A) = 2$$

$\because$ $\text{Rank}(A|B) \neq \text{Rank}(A)$

$\therefore$ Given system of equations has no solution for $\lambda = 6$ and $\mu \neq 20$.

Q.72 The system of linear equations

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 14 \end{bmatrix} \text{has}$$

(a) a unique solution
(b) infinitely many solutions
(c) no solution
(d) exactly two solutions

Solution : (b) [EC, GATE-2014 : 2 Marks, Set-2]

Q.73 Consider a system of linear equations:

$$x - 2y + 3z = -1,$$
$$x - 3y + 4z = 1, \text{ and}$$
$$-2x + 4y - 6z = k$$

The value of k for which the system has infinitely many solution is ________.

[EC, GATE-2015 : 1 Mark, Set-1]

Solution: (2)

$$x - 2y + 3z = -1,$$
$$x - 3y + 4z = 1, \text{ and}$$
$$-2x + 4y - 6z = k$$

$$[A : B] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 1 & -3 & 4 & 1 \\ -2 & 4 & 6 & k \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & k-2 \end{array}\right]$$

For infinite may solution

$$\rho(A : B) = \rho(A)$$
$$= r < \text{number of variables}$$
$$\rho(A : B) = 2$$
$$k - 2 = 0$$
$$k = 2$$

Q.74 Consider the following system of linear equations

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and the third columns of the coefficient matrix are linearly dependent. For how many values of α, does this system of equations have infinitely many solutions?

(a) 0
(b) 1
(c) 2
(d) infinitely many

[CS, GATE-2003, 2 marks]

Solution: (b)

The augmented matrix for the given system is $\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right]$.

Performing Gauss-Elimination on the above matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right] \xrightarrow[R_3 - 1/2 R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5 - 2\alpha \\ 0 & 3/2 & -6 & 7 - \alpha/2 \end{array}\right] \xrightarrow{R_3 - 3/2 R_2} \left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5 - 2\alpha \\ 0 & 0 & 0 & \frac{5\alpha - 1}{2} \end{array}\right]$$

Now for infinite solution it is necessary that at least one row must be completely zero.

$$\therefore \quad \frac{5\alpha - 1}{2} = 0$$
$$\alpha = 1/5 \text{ is the solution}$$

$\therefore$ There is only one value of α for which infinite solution exists.

Q.75 How many solutions does the following system of linear equations have?

$$-x + 5y = -1 \quad ; \quad x - y = 2 \quad ; \quad x + 3y = 3$$

(a) infinitely many
(b) two distinct solutions
(c) unique
(d) none

[CS, GATE-2004, 2 marks]

Solution: (c)

$$-x + 5y = -1$$
$$x - y = 2$$
$$x + 3y = 3$$

The avg mented matrix is $\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right]$.

Using gauss-elimination on above matrix we get,

$$\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+R_1} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Rank [A|B] = 2 (number of non zero rows in [A|B])

Rank [A] = 2 (number of non zero rows in [A])

Rank [A|B] = Rank [A] = 2 = number of variables

∴ Unique solution exists. Correct choice is (c).

Q.76 Consider the following system of equations in three real variables x_1, x_2 and x_3

$2x_1 - x_2 + 3x_3 = 1$

$3x_1 - 2x_2 + 5x_3 = 2$

$-x_1 - 4x_2 + x_3 = 3$

This system of equations has

(a) no solution

(b) a unique solution

(c) more than one but a finite number of solutions

(d) an infinite number of solutions

[CS, GATE-2005, 2 marks]

Solution: (b)

The augmented matrix for the given system is $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right]$.

Using gauss-elimination method on above matrix we get,

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right] \xrightarrow[R_3+\frac{1}{2}R_1]{R_2-\frac{3}{2}R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -1/2 & 1/2 & 1/2 \\ 0 & -9/2 & 5/2 & 7/2 \end{array}\right] \xrightarrow{R_3-9R_2} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

Rank ([A | B]) = 3

Rank ([A]) = 3

Since Rank ([A | B]) = Rank ([A]) = number of variables. The system has unique solution.

Q.77 The following system of equations

$x_1 + x_2 + 2x_3 = 1$; $x_1 + 2x_3 + 3x_3 = 2$; $x_1 + 4x_2 + ax_3 = 4$

has a unique solution. The only possible value(s) for a is / are

(a) 0

(b) either 0 or 1

(c) one of 0,1 or –1

(d) any real number other than 5

[CS, GATE-2008, 1 mark]

Solution: (d)

The augmented matrix for above system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 4 & a & 4 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & a-2 & 3 \end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & 0 \end{array}\right]$$

Now as long as $a - 5 \neq 0$, rank (A) = rank (A|B) = 3

∴ A can take any real value except 5. Closest correct answer is (d).

Q.78 Consider the following system of equations:

$$3x + 2y = 1$$
$$4x + 7z = 1$$
$$x + y + z = 3$$
$$x - 2y + 7z = 0$$

The number of solutions for this system is ________.

[CS, GATE-2014 : 1 Mark, Set-1]

Solution :

Given:

$$\begin{aligned} 3x + 2y &= 1 \\ 4x + 7z &= 1 \\ x + y + z &= 3 \\ x - 2y + 7z &= 0 \end{aligned}$$

$$\begin{aligned} x + y + z &= 3 \\ -x - 2y + 7z &= 0 \\ \hline 3y - 6z &= 3 \end{aligned}$$

$$\Rightarrow \quad y - 2z = 1$$

$$\Rightarrow \quad \begin{aligned} 2y - 4z &= 2 \\ 2y + 3x &= 1 \\ \hline 3x + 7z &= -1 \\ 4x + 7z &= 1 \\ \hline x &= 2 \end{aligned}$$

$$x = 2$$

$$3x + 2y = 1$$

$$\Rightarrow \quad y = -5/2$$

$$\Rightarrow \quad 4x + 7z = 1 \qquad \text{(Put } x = 2\text{)}$$

$$\begin{aligned} 8 + 7z &= 1 \\ \hline z &= -1 \end{aligned}$$

$\therefore$ The number of solutions for this system is one. $x = 2$, $y = -5/2$ and $z = -1$ is the only solution.

Q.79 If the following system has non-trivial solution,

$$px + qy + rz = 0$$
$$qx + ry + pz = 0$$
$$rx + py + qz = 0$$

then which one of the following options is TRUE?

(a) $p - q + r = 0$ or $p = q = -r$

(b) $p + q - r = 0$ or $p = -q = r$

(c) $p + q + r = 0$ or $p = q = r$

(d) $p - q + r = 0$ or $p = -q = -r$

[CS, GATE-2015 : 2 Marks, Set-3]

Solution: (c)

$$px + qy + rz = 0$$
$$qx + ry + pz = 0$$
$$rx + py + qz = 0$$

Let $A = \begin{bmatrix} p & q & r \\ q & r & p \\ r & p & q \end{bmatrix}$. The system is $A\hat{x} = O$

This is a homogenous system. Such a system has non-trivial solution iff |A|=0.

So, $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$p(qr - p^2) - q(q^2 - pr) + r(pq - r^2) = 0$

$p^3 + q^3 + r^3 - 3pqr = 0$

p = q = r satisfies the above equation.

Also if $p + q + r = 0$ then a can be transformed into one of the row as completely 0's as shown below.

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} \xrightarrow{R_1+R_2+R_3} \begin{vmatrix} p+q+r & p+q+r & p+q+r \\ q & r & p \\ r & p & q \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ q & r & p \\ r & p & q \end{vmatrix} = 0$$

Therefore the correct option is (c) which is $p + q + r = 0$ or $p = q = r$.

Q.80 Let A be an $n \times n$ matrix with rank $r(0 < r < n)$. Then $AX = 0$ has p independent solutions, where p is

(a) r
(b) n
(c) $n - r$
(d) $n + r$

[IN, GATE-2015 : 1 Mark]

Solution: (c)

Given
$$AX = 0$$
$$\rho(A_{n \times n}) = r\,(0 < r < n)$$

p = Number of independent solutions = nullity

We know that

$$\text{rank} + \text{nullity} = n$$
$$r + p = n$$
$$p = n - r$$

Q.81 Consider the following linear system.

$$x + 2y - 3z = a \;\;;\;\; 2x + 3y + 3x = b \;\;;\;\; 5x + 9y - 6z = c$$

This system is consistent if a, b and c satisfy the equation

(a) $7a - b - c = 0$
(b) $3a + b - c = 0$
(c) $3a - b + c = 0$
(d) $7a - b + c = 0$

[CE, 2016 : 2 Marks, Set-II]

Solution: (b)

We can represent the system of equation in matrix form as

$$\begin{bmatrix}1 & 2 & -3\\2 & 3 & 3\\5 & 9 & -6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$[A:B] = \begin{bmatrix}1 & 2 & -3 & : & a\\2 & 3 & 3 & : & b\\5 & 9 & -6 & : & c\end{bmatrix}$$

By elementary operation $R_3 \to R_3 - (3R_1 - R_2)$.

$$[A:B] = \begin{bmatrix}1 & 2 & -3 & : & a\\2 & 3 & 3 & : & b\\0 & 0 & 0 & : & c-3a-b\end{bmatrix}$$

For consisting of system, $c - 3a - b = 0$

Q.82 The solution to the system of equations is $\begin{bmatrix}2 & 5\\-4 & 3\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix} = \begin{Bmatrix}2\\-30\end{Bmatrix}$

(a) 6, 2
(b) –6, 2
(c) –6, –2
(d) 6, – 2

[ME, 2016 : 1 Mark, Set-1]

Solution: (d)

$$\begin{bmatrix}2 & 5\\-4 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\-30\end{bmatrix}$$

$$\begin{bmatrix}2 & 5 & 2\\-4 & 3 & -30\end{bmatrix}$$

$$R_2 + 2R_1$$

$$\begin{bmatrix}2 & 5 & 2\\0 & 13 & -26\end{bmatrix}$$

$$13y = -26$$

or $$y = -2$$

$$2x + 5y = 2$$

$$2x + 5(-2) = 2$$

$$2x = 2 + 10$$

$$2x = 12$$

or $$x = 6$$

Q.83 Consider the systems, each consisting of m linear equations in n variables.

I. If $m < n$, then all such systems have a solution.

II. If $m > n$, then none of these sytems has a solution.

III. If $m = n$, then there exists a system which has a solution

Which one of the following is CORRECT?

(a) I, II and III are true
(b) Only II and III are true
(c) Only III is true
(d) None of them is true

[CS, 2016 : 1 Mark, Set-2]

Solution: (c)

I. $m < n$ (system may still be inconsistent so incorrect)

II. $m > n$ (rank may still be equal to n of hence solution may exist so incorrect).

III. $m = n$ (some system rank may be equal to n and hence may have solution so correct).

So only III is correct.

1.8 EIGENVALUES AND EIGENVECTORS

Let $A = [a_{ij}]_{n \times n}$ be any n-rowed square matrix and λ is a scalar. The equation $AX = \lambda X$ is called eigen value problem. We wish to find non zero solutions to X satisfying the eigen value problem, and these non zero solution to X are called as the **eigen vectors** of A. The corresponding λ values are called **eigen values** of A.

1.8.1 Definitions

The matrix $A - \lambda I$ is called **characteristic matrix** of A, where I is the unit matrix of order n. Also the determinant

$$|A - \lambda I| = \begin{bmatrix} a_{11-\lambda} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22-\lambda} \cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ a_{n1} & a_{n2} \cdots\cdots a_{nn-\lambda} \end{bmatrix}$$

which is ordinary polynomial in λ of degree n is called **"characteristic polynomial of A"**. The equation $|A - \lambda I| = 0$ is called **"characteristic equation of A"**.

Characteristic Roots: The roots of the characteristic equation are called **"characteristic roots or characteristic values or latent roots or proper values or eigen values"** of the matrix A. The set of eigenvalues of A is called the **"spectrum of A"**.

If I is a characteristic root of the matrix A, then if $|A - \lambda I| = 0$, then the matrix $A - \lambda I$ is singular. Therefore there exist a non-zero vector X such that $(A - \lambda I)X = 0$ or $AX = \lambda X$, which is the eigen value problem.

Characteristic Vectors: If λ is a characteristic root of an $n \times n$ matrix A, then a non-zero vector X such that $AX = \lambda X$ is called characteristic vector or eigenvector of A corresponding to characteristic root λ.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.84 The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

(a) are 1 and 4
(b) are −1 and 2
(c) are 0 and 5
(d) cannot be determined

[CE, GATE-2004, 2 marks]

Solution: (c)

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda) \times (1-\lambda) - [(-2) \times (-2)] = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\Rightarrow \quad \lambda(\lambda - 5) = 0$$

Hence, $\lambda = 0, 5$ are the eigen values.

Q.85 The Eigen values of the matrix $[P] = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$ are

(a) –7 and 8
(b) –6 and 5
(c) 3 and 4
(d) 1 and 2

[CE, GATE-2008, 2 marks]

Solution: (b)

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 5 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (4-\lambda)(-5-\lambda)-2\times 5 = 0$$

$$\Rightarrow \quad \lambda^2+\lambda-30 = 0$$

$$\lambda = 5, -6$$

Q.86 The smallest and largest Eigen values of the following matrix are

$$\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$$

(a) 1.5 and 2.5
(b) 0.5 and 2.5
(c) 1.0 and 3.0
(d) 1.0 and 2.0

[CE, GATE-2015 : 2 Marks, Set-I]

Solution: (d)

For eigen values $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 2 \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-20+4\lambda-5\lambda+\lambda^2+18)+2\,(20-4\lambda-12)+2\,(-12+8+2\lambda) = 0$$

$$\Rightarrow \quad \lambda^3-4\lambda^2+5\lambda-2 = 0$$

Only 1 and 2 satisfy this equation.

$\lambda = 1, 1, 2$

Hence,Smallest eigen value = 1 and

Largest eigen value = 2

Q.87 For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ the eigen value are

(a) 3 and – 3
(b) –3 and – 5
(c) 3 and 5
(d) 5 and 0

[ME, GATE-2003, 1 mark]

Solution: (c)

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Now, $A - \lambda I = 0$

Where λ = eigen value

$$\therefore \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)^2 - 1 = 0$$

or, $(4-\lambda)^2 - (1)^2 = 0$

or, $(4-\lambda+1)(4-\lambda-1) = 0$

or, $(5-\lambda)(3-\lambda) = 0$

$\therefore$ $\lambda = 3, \lambda = 5$

Q.88 Which one of the following is an eigenvector of the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

[ME, GATE-2005, 2 marks]

Solution: (a)

First solve for eigen values by solving characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$= (5-\lambda)(5-\lambda)[(2-\lambda)(1-\lambda) - 3] = 0$$

$$= (5-\lambda)(5-\lambda)(\lambda^2 - 3\lambda - 1) = 0$$

$$\lambda = 5, 5, \frac{3 \pm \sqrt{13}}{2}$$

put $\lambda = 5$ in $[A - \lambda I]\hat{X} = 0$

$$\begin{bmatrix} 5-5 & 0 & 0 & 0 \\ 0 & 5-5 & 5 & 0 \\ 0 & 0 & 2-5 & 1 \\ 0 & 0 & 3 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 5x_3 = 0$$

$$-3x_3 + x_4 = 0$$

$$3x_3 - 4x_4 = 0$$

Solving which we get $x_3 = 0$, $x_4 = 0$, x_1 and x_2 may be anything.

The eigen vector corresponding to $\lambda = 5$, may be written as

$$\hat{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

where k_1, k_2 may be any real number. Since choice (a) is the only matrix in this form with both x_3 and $x_4 = 0$, so it is the correct answer.

Since, we already got a correct eigen vector, there is no need to derive the eigen vector corresponding to $\lambda = \frac{3 \pm \sqrt{13}}{2}$.

Q.89 The number of linearly independent eigenvectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

(a) 0
(b) 1
(c) 2
(d) infinite

[ME, GATE-2007, 2 marks]

Solution: (b)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$[a - \lambda I] = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)^2 = 0$$

$$\Rightarrow \quad \lambda = 2$$

Now, consider the eigen value problem

$$[A - \lambda I]\hat{X} = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$, we get, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_2 = 0 \quad \ldots (i)$$

$$0 = 0 \quad \ldots (ii)$$

The solution is therefore $x_2 = 0$, x_1 = anything

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Q.90 The eigenvectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is a + b ?

(a) 0 (b) 1/2
(c) 1 (d) 2 **[ME, GATE-2008, 2 marks]**

Solution: (b)

$$\begin{vmatrix} (1-\lambda) & 2 \\ 0 & (2-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda) = 0$$

$$\therefore \quad \lambda = 1, 2$$

Now since the eigen value problem is

$$[A - \lambda I]\hat{X} = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\hat{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

putting the value of $\lambda = 1$ and $\hat{X} = \hat{X}_1 = \begin{bmatrix} 1 \\ a \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 0$$

$$\Rightarrow \quad a = 0 \qquad \text{...(i)}$$

putting the value of $\lambda = 2$ and $\hat{X} = \hat{X}_2 = \begin{bmatrix} 1 \\ b \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 0$$

$$\Rightarrow \quad -1 + 2b = 0$$

and $\quad 0 = 0$

$$\Rightarrow \quad b = \frac{1}{2} \qquad \text{...(ii)}$$

From (i) and (ii) $\quad a + b = 0 + \frac{1}{2} = \frac{1}{2}$

Q.91 One of the eigen vectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is

(a) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (b) $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$
(c) $\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ **[ME, GATE-2010, 2 marks]**

Solution: (a)

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(3-\lambda)-2 = 0$$
$$\lambda^2-5\lambda+4 = 0$$
$$\lambda = 1, 4$$

The eigen value problem is $[A-\lambda I]\hat{x} = 0$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 1$, $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 = 0 \quad \text{...(i)}$$
$$x_1 + 2x_2 = 0 \quad \text{...(ii)}$$

Solution is $x_2 = k$, $x_1 = -2k$

$$\hat{X}_1 = \begin{bmatrix} -2k \\ k \end{bmatrix} \text{ i.e. } x_1 : x_2 = -2 : 1$$

Since, choice (A) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in same ratio of x_1 to x_2.

$\therefore$ Choice (a) is an eigen vector.

Q.92 The lowest eigenvalue of the 2 × 2 matrix $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is ______.

[ME, GATE-2015 : 1 Mark, Set-3]

Solution: (2)

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$\therefore$ $|A-\lambda I| = 0$

$$\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda)-2 = 0$$

$$(\lambda-4)(\lambda-3)-2 = 0$$
$$\lambda^2-7\lambda+10 = 0$$

$\Rightarrow$ $\lambda = 5, 2$

Miniumum value = 2

Q.93 For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen values is equal to –2. Which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

[EE, GATE-2005, 2 marks]

Solution: (d)

Since matrix is triangular, the eigen values are the diagonal elements themselves namely $\lambda = 3, -2$ and 1. Corresponding to eigen value, $\lambda = -2$ let us find the eigen vector

$$[A - \lambda I]\,\hat{x} = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -2$ in above equation we get,

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations,

$$5x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$
$$x_3 = 0 \quad \text{... (ii)}$$
$$3x_3 = 0 \quad \text{... (iii)}$$

Since eq. (ii) and (iii) are same we have

$$5x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$
$$x_3 = 0 \quad \text{... (ii)}$$

Putting $x_2 = k$, in eq. (i) we get

$$5x_1 - 2k + 2 \times 0 = 0$$

$\Rightarrow$ $x_1 = 2/5\,k$

$\therefore$ Eigen vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/5\,k \\ k \\ 0 \end{bmatrix}$$

i.e. $x_1 : x_2 : x_3 = 2/5\,k : k : 0 = 2/5 : 1 : 0 = 2 : 5 : 0$

$\therefore$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$ is an eigen vector of matrix p.

Q.94 The linear operation L(x) is defined by the cross product $L(x) = b \times X$, where $b = [0\ 1\ 0]^T$ and $X = [x_1\ x_2\ x_3]^T$ are three dimensional vectors. The 3 × 3 matrix M of this operation satisfies

$$L(x) = M\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then the eigenvalues of M are

(a) 0, +1, −1
(b) 1, −1, 1
(c) i, −i, 1
(d) i, −i, 0

[EE, GATE-2007, 2 marks]

Solution: (d)

The cross product of $b = [0 \quad 1 \quad 0]^t$ and $X = [x_1 \quad x_2 \quad x_3]^t$ can be written as

$$b \times X = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} = x_3\,\hat{i} + 0\,\hat{j} - x_1\,\hat{k}$$

$$= [x_3 \quad 0 \quad -x_1]$$

Now $$L(x) = b \times X = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where M is a 3 × 3 matrix

Let $$M = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

Now $$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b \times X$$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

By matching LHS and RHS we get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

So, $$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Now we have to find the eigen values of M

$$|M - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 0) + 1(0 - \lambda) = 0$$

$$\Rightarrow \lambda^3 + \lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 + 1) = 0$$

$$\Rightarrow \lambda = 0, \lambda = \pm i$$

So, the eigen values of M are i, –i and 0.

Correct choice is (d).

Q.95 An eigenvector of $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) $[-1 \quad 1 \quad 1]^T$
(b) $[1 \quad 2 \quad 1]^T$
(c) $[1 \quad -1 \quad 2]^T$
(d) $[2 \quad 1 \quad -1]^T$

[EE, GATE-2010, 2 marks]

Solution: (b)

Given, $$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

P is triangular. So eigen values are the diagonal elements themselves. Eigen values are therefore, $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

Now, the eigen value problem is $[A - \lambda I]\,\hat{x} = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda_1 = 1$, we get the eigen vector corresponding to this eigen value,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations

$$x_2 = 0$$
$$x_2 + 2x_3 = 0$$
$$2x_3 = 0$$

The solution is $x_2 = 0, x_3 = 0, x_1 = k$

So, one eigen vector is $\hat{x}_1 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ i.e., $x_1 : x_2 : x_3 = k : 0 : 0$

Since, none of the eigen vectors given in choices matches with this, ratio we need to proceed further and find the other eigen vectors corresponding to the other Eigen values.

Now, corresponding to $\lambda_2 = 2$, we get by substituting $\lambda = 2$, in the eigen value problem, the following set of equations,

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations,

$$-x_1 + x_2 = 0$$
$$2x_3 = 0$$
$$x_3 = 0$$

Solution is $x_3 = 0, x_1 = k, x_2 = k$

$$\therefore \quad \hat{X}_2 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} \text{ i.e., } x_1 : x_2 : x_3 = 1 : 1 : 0$$

Since none of the eigen vectors given in the choices is of this ratio, we need to proceed further and find 3rd eigen vector also.

By putting $\lambda = 3$ in the eigen valve problem, we get

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 = 0$$
$$-x_2 + 2x_3 = 0$$

putting $x_1 = k$, we get, $x_2 = 2k$ and $x_3 = x_2/2 = k$

$$\therefore \quad \hat{X}_3 = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} \text{ i.e., } x_1 : x_2 : x_3 = 1 : 2 : 1$$

Only the eigen vector given in choice (b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, is in this ratio. So, the correct answer is choice (b).

Q.96 One pair of eigen vectors corresponding to the two eigen values of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$

[IN, GATE-2013 : 2 marks]

Solution: (a, d)

Eigen values are

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$
$$\lambda^2 = -1$$
$$\therefore \quad \lambda = \pm i$$

to find eigen vector,

$\lambda = +i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $$-i\,x_1 - x_2 = 0 \text{ and } x_1 - ix_2 = 0$$

clearly, $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -j \end{bmatrix} \text{and} \begin{bmatrix} j \\ 1 \end{bmatrix}, \text{ satisfy}$$

$\lambda = -i$ $$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$ix_1 - x_2 = 0 \text{ and } x_1 + ix_2 = 0$$

clearly,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} j \\ -1 \end{bmatrix} \text{and} \begin{bmatrix} 1 \\ j \end{bmatrix}, \text{ satisfy}$$

Thus, the two eigen value of the given matrix are $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$.

Q.97 Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigenvector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Solution: (c)

First, find the eigen values of $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-4-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 20 = 0$$

$$\Rightarrow (\lambda + 5)(\lambda - 4) = 0$$

$$\Rightarrow \lambda_1 = -5 \text{ and } \lambda_2 = 4$$

Corresponding to $\lambda_1 = -5$ we need to find eigen vector:

The eigen value problem is $[A - \lambda I]\hat{X} = 0$

$$\Rightarrow \begin{bmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0$$

Putting $\lambda = -5$

we get, $$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad \text{...(i)}$$

$$4x_1 + 8x_2 = 0 \quad \text{...(ii)}$$

Since (i) and (ii) are the same equation we take

$$x_1 + 2x_2 = 0 \quad \text{...(i)}$$

$$x_1 = -2x_2$$
$$x_1 : x_2 = -2 : 1$$
$$\Rightarrow \quad \frac{x_1}{x_2} = -2$$

Now from the answers given, we look for any vector in this ratio and we find choice (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in this ratio $\frac{x_1}{x_2} = \frac{2}{-1} = -2$.

So choice (c) is an eigen vector corresponding to $\lambda = -5$.

Since we already got an answer, there is no need to find the second eigen vector corresponding to $\lambda = 4$.

Q.98 For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(a) 2 (b) 4
(c) 6 (d) 8

[EC, GATE-2006, 2 marks]

Solution: (c)

$$M = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, [M - \lambda I] = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

Given eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$

$$[M - \lambda I]\hat{X} = 0$$
$$\Rightarrow \quad \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix} = 0$$
$$\Rightarrow \quad (4-\lambda)(101) + 2 \times 101 = 0$$
$$\Rightarrow \quad \lambda = 6$$

Q.99 The value of p such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$ is

[EC, GATE-2015 : 1 Mark, Set-1]

Solution:

$$AX = \lambda X$$
$$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} 12 \\ p+7 \\ 36 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$\frac{p+7}{12} = 2 \quad \Rightarrow p = 17$$

Q.100 What are the eigenvalues of the following 2 × 2 matrix?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

(a) −1 and 1
(b) 1 and 6
(c) 2 and 5
(d) 4 and −1 **[CS, GATE-2005, 2 marks]**

Solution: (b)

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

The characteristic equation of this matrix is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda = 1, 6$$

∴ The eigen values of A are 1 and 6.

Q.101 How many of the following matrices have an eigenvalue 1?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(a) one
(b) two
(c) three
(d) four **[CS, GATE-2008, 2 marks]**

Solution: (a)

Eigen values of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) * -\lambda = 0$$

$$\lambda = 0 \text{ or } \lambda = 1$$

Eigen values of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 = 0$$

$$\lambda = 0, 0$$

Eigen values of $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)^2 + 1 = 0$$

$$(1-\lambda)^2 = -1$$

$$1-\lambda = i \text{ or } -i$$

$$\lambda = 1 - i \text{ or } 1 + i$$

Eigen values of $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix}$

$$(-1-\lambda)(-1-\lambda) = 0$$
$$(1+\lambda)^2 = 0$$
$$\lambda = -1, -1$$

So, only one matrix has an eigen value of 1 which is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
Correct choice is (a).

Q.102 Let A be the 2 × 2 matrix with elements $a_{11} = a_{12} = a_{21} = +1$ and $a_{22} = -1$. Then the eigenvalues of the matrix A^{19} are

(a) 1024 and – 1024
(b) $1024\sqrt{2}$ and $-1024\sqrt{2}$
(c) $4\sqrt{2}$ and $-4\sqrt{2}$
(d) $512\sqrt{2}$ and $-512\sqrt{2}$

[CS, GATE-2012, 1 marks]

Solution: (d)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigen (A) are the roots of the characteristic polynomial given below:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(-1-\lambda) - 1 = 0$$
$$-(1-\lambda)(1+\lambda) - 1 = 0$$
$$\lambda^2 - 2 = 0$$
$$\lambda = \pm\sqrt{2}$$

Eigen values of A are $\sqrt{2}$ and $-\sqrt{2}$ respectively.

So eigen values of $A^{19} = (\sqrt{2})^{19}$ and $(-\sqrt{2})^{19}$

$$= 2^{19/2} \text{ and } -2^{19/2}$$
$$= 2^9.2^{1/2} \text{ and } -2^9.2^{1/2}$$
$$= 512\sqrt{2} \text{ and } -512\sqrt{2}.$$

Q.103 The larger of the two eigenvalues of the matrix $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ is ______.

[CS, GATE-2015 : 1 Mark, Set-2]

Solution: (6)

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 4-\lambda & 5 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 10 = 0$$
$$\lambda^2 - 5\lambda - 6 = 0$$
$$(\lambda - 6)(\lambda + 1) = 0$$
$$\Rightarrow \quad \lambda = 6, -1$$

∴ Maximum eigen value is '6'.

Q.104 In the given matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, one of the eigenvalues is 1. The eigenvectors corresponding to the eigenvalue 1 are

(a) $\{\alpha(4,2,1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$
(b) $\{\alpha(-4,2,1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$
(c) $\{\alpha(\sqrt{2},0,1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$
(d) $\{\alpha(-\sqrt{2},0,1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$

[CS, GATE-2015 : 1 Mark, Set-3]

Solution: (b)

Let $$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Given eigen value $\lambda = 1$.

Lex X be the vector. Then $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & -1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{bmatrix} X = 0$$

put $\lambda = 1$

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -x_2 + 2x_3 \\ 0 \\ x_1 + 2x_2 \end{bmatrix} = 0$$

putting $x_1 = k$ we get $x_2 = -k/2$ and $x_3 = -k/4$

So the eigen vector $= k\begin{bmatrix} 1 \\ -1/2 \\ -1/4 \end{bmatrix}$

The ratios are $x_1/x_2 = \dfrac{-1}{-1/2} = -2$ and $x_2/x_3 = \dfrac{-1/2}{-1/4} = 2$

Only option (b) (–4, 2, 1) has the same ratios and therefore is a correct eigen vector.

Q.105 If the entries in each column of a square matrix M add up to 1, then an eigen value of M is

(a) 4
(b) 3
(c) 2
(d) 1

[CE, 2016 : 1 Mark, Set-I]

Solution: (d)

Consider the '2 × 2' square matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\Rightarrow \quad \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \quad \text{...(i)}$$

Putting $\lambda = 1$, we get

$1 - (a + d) + ad - bc = 0$

$1 - a - d + ad - (1 - d)(1 - a) = 0$

$1 - a - d + ad - 1 + a + d - ad = 0$

$0 = 0$ which is true.

$\therefore$ $\lambda = 1$ satisfied the eq. (i) but $\lambda = 2, 3, 4$ does not satisfy the eq. (i). For all possible values of a, d.

Q.106 Consider a 3 × 3 matrix with every element being equal to 1. Its only non-zero eigenvalue is ______.

[EE, 2016 : 1 Mark, Set-1]

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Eigen value are 0, 0, 3

Q.107 The condition for which the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive, is

(a) $k > \frac{1}{2}$

(b) $k > -2$

(c) $k > 0$

(d) $k < -\frac{1}{2}$

[ME, 2016 : 1 Mark, Set-2]

Solution: (a)

All Eigen values of $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive

$$2 > 0$$

$\therefore$ 2 × 2 leading minor must be greater than zero

$$\begin{vmatrix} 2 & 1 \\ 1 & k \end{vmatrix} > 0$$

$$2k - 1 > 0$$

$$2k > 1$$

$$k > \frac{1}{2}$$

Q.108 Consider a 2 × 2 square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

where x is unknown. If the eigenvalues of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, then x is equal to

(a) $+j\omega$

(b) $-j\omega$

(c) $+\omega$

(d) $-\omega$

[EC, 2016 : 1 Mark, Set-3]

Solution: (d)

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

$$\text{Trace} = \text{sum of eigen values}$$

$$2\sigma = \sigma + j\omega + \sigma - j\omega$$

$$|A| = \text{product of eigens}$$

$$\sigma^2 - x\omega = (\sigma + j\omega)(\sigma - j\omega) = \sigma^2 + \omega^2$$

which is possible only when $x = -\omega$

Q.109 Two eigenvalues of a 3 × 3 real matrix P are $(2 + \sqrt{-1})$ and 3. The determinant of P is ______.

[CS, 2016 : 1 Mark, Set-1]

Solution:

Two eigen values are $2 + i$ and 3 of a 3 × 3 matrix. The third eigen value must be $2 - i$

Now $\Pi\lambda_i = |A|$

$\Rightarrow$ $|A| = (2 + i)(2 - i) \times 3 = (4 - i^2) \times 3$

$= 5 \times 3 = 15$

1.8.2 Some Results Regarding Characteristic Roots and Characteristic Vectors

1. λ is a characteristic root of a matrix A iff there exist a non-zero vector X such that $AX = \lambda X$.
2. If X is a characteristic vector of matrix A corresponding to characteristic value λ, then kX is also a characteristic vector of A corresponding to the same characteristic value λ where k is non-zero vector.
3. If X is a characteristic vector of a matrix A, then X cannot correspond to more than one characteristic values of A.
4. If a matrix A is of size n × n, and if it has n distinct eigen values, then there will be n linearly independent eigen vectors. However, if the n eigen values are not distinct, then there may or may not be n linearly independent eigen vectors.
5. The characteristic roots (Eigen values) of a Hermitian matrix are real.
6. The characteristic roots (Eigen values) of a real symmetric matrix are all real, since every such matrix is Hermitian.
7. Characteristic roots (Eigen values) of a skew Hermitian matrix are either pure imaginary or zero.
8. The characteristic roots (Eigen values) of a real skew symmetric matrix are either pure imaginary or zero, for every such matrix is skew Hermitian.
9. The characteristic roots (Eigen values) of a unitary matrix are of unit modulus i.e., $|\lambda| = 1$.
10. The characteristic roots (Eigen values) of an orthogonal matrix is also of unit modulus, since every such matrix is unitary.

1.8.3 Process of Finding the Eigenvalues and Eigenvectors of a Matrix

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n, first we should write the characteristic equation of the matrix A. i.e., the equation $|A - \lambda I| = 0$. This equation will be of degree n in λ. So it will have n roots. These n roots will be the n eigenvalues of the matrix A.

If λ_1 is an eigenvalue of A, the corresponding eigenvectors of A will be given by the non-zero vectors $X_1 = [x_1, x_2, \ldots\ldots\ldots\ldots x_n]'$ satisfying the equations $AX_1 = \lambda_1 X_1$ or $[A - \lambda_1 I]X_1 = 0$

1.8.4 Properties of Eigen Values

1. If $\lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_n$ are the eigenvalues of A, then $k\lambda_1, k\lambda_2 \ldots\ldots\ldots k\lambda_n$ are eigenvalues of kA.
2. the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

   i.e. if $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are two eigen value of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigen value of A^{-1}.
3. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$ are the eigen values of A^k.
4. If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the eigen values of a non-singular matrix A, then $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2} \ldots \frac{|A|}{\lambda_n}$ are the eigen values of Adj A.
5. Eigen values of A = Eigen values of A^T.
6. Maximum no. of distinct eigen values = size of A.
7. Sum of eigen values = Trace of A = Sum of diagonal elements.
8. Product of eigen values = $|A|$ (i.e. At least one eigen value is zero iff A is singular).
9. In a triangular and diagonal matrix, eigen values are diagonal elements themselves.
10. Similar matrices have same eigen values. Two matrices A and B are said to be similar if there exists a non singular matrix P such that $B = P^{-1}AP$.
11. If A and B are two matrices of same order then the matrix AB and BA will have same characteristic roots.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.110 Consider the system of equations $A_{(n \times n)} x_{(n \times 1)} = \lambda_{(n \times 1)}$ where, λ is a scalar. Let (λ_i, x_i) be an eigen-pair of an eigen value and its corresponding eigen vector for real matrix A. Let I be a $(n \times n)$ unit matrix. Which one of the following statement is NOT correct?

(a) For a homogeneous $n \times n$ system of linear equations, $(A - \lambda I)x = 0$ having a nontrivial solution, the rank of $(A - \lambda I)$ is less than n

(b) For matrix A^m, m being a positive integer, (λ_i^m, x_i^m) will be the eigen-pair for all i

(c) If $A^T = A^{-1}$, then $|\lambda_i| = 1$ for all i

(d) If $A^T = A$, then λ_i is real for all i

[CE, GATE-2005, 2 marks]

Solution: (b)

Although λ_i^m will be the corresponding eigen values of A^m, x_i^m need not be corresponding eigen vectors.

Q.111 For a given matrix

$$A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix},$$

one of the eigenvalues is 3. The other two eigenvalues are

(a) 2, −5
(b) 3, −5
(c) 2, 5
(d) 3, 5

[CE, GATE-2006, 2 marks]

Solution: (b)

$$\Sigma\lambda_i = \text{Trace (A)}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace (A)} = 2 + (-1) + 0 = 1$$

Now $\lambda_1 = 3$

$\therefore$ $3 + \lambda_2 + \lambda_3 = 1$

$\Rightarrow$ $\lambda_2 + \lambda_3 = -2$

Only choice (b) satisfies this condition.

Q.112 The minimum and the maximum eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are −2 and 6, respectively. What is the other eigen value?

(a) 5
(b) 3
(c) 1
(d) −1

[CE, GATE-2007, 1 mark]

Solution: (b)

$$\Sigma\lambda_i = \text{Trace (A)}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1 = 7$$

Now $\lambda_1 = -2, \lambda_2 = 6$

$\therefore$ $-2 + 6 + \lambda_3 = 7$

$$\lambda_3 = 3$$

Q.113 The sum of Eigen values of matrix, [M] is

$$\text{where } [M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$$

(a) 915
(b) 1355
(c) 1640
(d) 2180

[CE, GATE-2014 : 1 Mark, Set-I]

Solution : (a)

Sum of eigen values = trace of matrix
= 215 + 150 + 550 = 915

Q.114 The two Eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3 : 1 for $p = 2$. What is another value of p for which the Eigen values have the same ratio of 3 : 1?

(a) −2
(b) 1
(c) 7/3
(d) 14/3

[CE, GATE-2015 : 2 Marks, Set-II]

Solution: (d)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$$

Let λ_1 and λ_2 be the eigen value of matrix A

$\therefore$ $\dfrac{\lambda_1}{\lambda_2} = \dfrac{3}{1}$ for $p = 2$

Sum of eigen value

$$= \lambda_1 + \lambda_2 = 2 + p \quad \text{...(i)}$$

Product of eigen value

$$= \lambda_1\lambda_2 = 2p - 1 \quad \text{...(ii)}$$

$$\frac{\lambda_1}{\lambda_2} = \frac{3}{1}$$

$\Rightarrow$ $$\lambda_1 = 3\lambda_2$$

From eq. (i),

$\Rightarrow$ $$3\lambda_2 + \lambda_2 = 2 + p$$

$$4\lambda_2 = 2 + p$$

$$\lambda_2 = \frac{p+2}{4}$$

From eq. (ii)

$\Rightarrow$ $$3\lambda_2^2 = 2p - 1$$

$\Rightarrow$ $$3\left(\frac{P+2}{4}\right)^2 = 2p - 1$$

$\Rightarrow$ $$p = 2, \frac{14}{3}$$

OR

$$\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$$

$$A - I\lambda = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & p-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(p-\lambda) - 1 = 0$$

$$\lambda^2 - (p + Z)\lambda + (2p - 1) = 0$$

By putting values of p from options.

By putting option (d) $\frac{14}{3}$ in above equations gives value $5, \frac{5}{3}$.

Hence ratio of two eigen values $= \frac{5}{5/3} = 3:1$.

So option (d) is correct.

Q.115 The sum of the eigen values of the matrix given below is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) 5
(b) 7
(c) 9
(d) 18

Solution: (b) **[ME, GATE-2004, 1 mark]**

Sum of eigen values of given matrix = sum of diagonal element of given matrix = $1 + 5 + 1 = 7$.

Q.116 Eigenvalues of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1. What are the eigenvalues of the matrix $S^2 = SS$?

(a) 1 and 25
(b) 6 and 4
(c) 5 and 1
(d) 2 and 10 **[ME, GATE-2006, 2 marks]**

Solution: (a)

If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_4$ are the eigen values of A. Then the eigen values of

A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m \ldots$

Here, S matrix has eigen values 1 and 5.

So, S^2 matrix has eigen values 1^2 & 5^2 i.e. 1 and 25.

Q.117 If a square matrix A is real and symmetric, then the eigenvalues

(a) are always real
(b) are always real and positive
(c) are always real and non-negative
(d) occur in complex conjugate pairs

[ME, GATE-2007, 1 mark]

Solution: (a)

The eigen values of any symmetric matrix is always real.

Q.118 The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has one eigenvalue equal to 3. The sum of the other two eigenvalues is

(a) p
(b) p −1
(c) p −2
(d) p −3 **[ME, GATE-2008, 1 mark]**

Solution: (c)

Sum of the eigen values of matrix is = trace of matrix = sum of diagonal values present in the matrix

$$\therefore \quad 1 + 0 + p = 3 + \lambda_2 + \lambda_3$$
$$\Rightarrow \quad p + 1 = 3 + \lambda_2 + \lambda_3$$
$$\Rightarrow \quad \lambda_2 + \lambda_3 = p + 1 - 3 = p - 2$$

Q.119 Eigenvalues of a real symmetric matrix are always

(a) positive
(b) negative
(c) real
(d) complex **[ME, GATE-2011, 1 mark]**

Solution: (c)

Eigen values of symmetric matrix are always real.

Q.120 For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, ONE of the normalized eigen vectors is given as

(a) $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$
(b) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$
(c) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$
(d) $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ **[ME, GATE-2012, 2 marks]**

Solution: (b)

$$A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation is

$$\begin{vmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(3-\lambda) - 3 = 0$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$\lambda = 2, 6$$

Now, to find eigen vectors:

$$[A - \lambda I]\hat{x} = 0$$

Which is $\begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Put $\lambda = 2$ in above equation and we get

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives us the equation,

$$3x_1 + 3x_2 = 0$$

and

$$x_1 + x_2 = 0$$

Which is only one equation,

$$x_1 + x_2 = 0$$

Whose solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix}$$

So one eigen vector is $\hat{x}_1 = \begin{bmatrix} -k \\ k \end{bmatrix}$

Which after normalization is $= \dfrac{\hat{x}_1}{|\hat{x}_1|}$

$$= \frac{1}{\sqrt{(-k)^2 + (k^2)}}\begin{bmatrix} -k \\ k \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

The other eigen vector is obtained by putting the other eigen value $\lambda = 6$ in eigen value problem

$$\begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives,

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives the equation

$$-x_1 + 3x_2 = 0$$

and

$$x_1 - 3x_2 = 0$$

Which is only one equation

$$-x_1 + 3x_2 = 0$$

Whose solution is

$$\hat{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3k \\ k \end{bmatrix}$$

Which after normalization is

$$\frac{\hat{x}_2}{|\hat{x}_2|} = \frac{1}{\sqrt{((3k)^2 + k^2)}} \begin{bmatrix} 3k \\ k \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Choice (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is the only correct choice,

since it is a constant multiple of one the normalized vectors which is $\hat{x}_1$.

Q.121 The eigen values of a symmetric matrix are all

(a) complex with non-zero positive imaginary part

(b) complex with non-zero negative imaginary part

(c) real

(d) pure imaginary

[ME, GATE-2013, 1 Mark]

Solution: (c)

(i) The Eigen values of symmetric matrix $[A^T = A]$ are purely real.

(ii) The Eigen value of skew-symmetric matrix $[A^T = -A]$ are either purely imaginary or zeros.

Q.122 Consider a 3 × 3 real symmetric matrix S such that two of its eigenvalues are $a \neq 0$, $b \neq 0$ with respective eigenvectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. If $a \neq b$ then $x_1y_1 + x_2y_2 + x_3y_3$ equals

(a) a

(b) b

(c) ab

(d) 0

[ME, GATE-2014 : 1 Mark, Set-3]

Solution : (d)

3 × 3 real symmetric matrix such that two of its eigen value are $a \neq 0$ $b \neq 0$ with respective eigen vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ if $a \neq b$ then

$x_1y_1 + x_2y_2 + x_3y_3 = 0$ because they are orthogonal.

$$\therefore \quad x^T y = 0 \qquad \text{(since } a \neq b\text{)}$$

$$[x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

Q.123 One of the eigenvectors of matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is

(a) $\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ (b) $\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$

(c) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

[ME, GATE-2014 : 1 Mark, Set-2]

Solution : (d)

The characteristic equation $|A - \lambda I| = 0$

i.e. $$\begin{vmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{vmatrix} = 0$$

or $$(\lambda - 6)(\lambda + 5) + 18 = 0$$

or $$\lambda^2 - 6\lambda + 5\lambda - 30 + 18 = 0$$

or $$\lambda^2 - \lambda - 12 = 0$$

or $$\lambda = \frac{1 \pm \sqrt{1+48}}{2} = \frac{1 \pm 7}{2} = 4, -3$$

Corresponding to $\lambda = 4$, we have

$$[A - \lambda I]x = \begin{bmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 0$$

or, $$\begin{bmatrix} -9 & 2 \\ -9 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives only one independent equation, $-9x + 2y = 0$

$\therefore \frac{x}{2} = \frac{y}{9}$ gives eigen vector (2, 9)

Corresponding to $\lambda = -3$,

$$= \begin{bmatrix} -2 & 2 \\ -9 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives $-x + y = 0$ (only one independent equation)

$\therefore$ $$\frac{x}{1} = \frac{y}{1}$$ which gives (1, 1)

So, the eigen vectors are $\begin{Bmatrix} 2 \\ 9 \end{Bmatrix}$ and $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$.

Q.124 At least one eigenvalue of a singular matrix is

(a) positive (b) zero

(c) negative (d) imaginary

[ME, GATE-2015 : 1 Mark, Set-3]

Solution: (b)

For singular matrix

$$|A| = 0$$

According to properties of eigen value

Product of eigen values = $|A| = 0$

$\Rightarrow$ Atleast one of the eigen value is zero.

Q.125 The trace and determinant of a 2 × 2 matrix are known to be –2 and –35 respectively. It eigenvalues are

(a) –30 and –5
(b) –37 and –1
(c) –7 and 5
(d) 17.5 and –2

[EE, GATE-2009, 1 mark]

Solution: (c)

$$\Sigma \lambda_i = \text{Trace (A)} = -2 \Rightarrow \lambda_1 + \lambda_2 = -2 \quad \text{... (i)}$$

$$\Pi \lambda_i = |A| = -35 \Rightarrow \lambda_1 \lambda_2 = -35 \quad \text{... (ii)}$$

Solving (i) and (ii) we get λ_1 and $\lambda_2 = 5, -7$.

Q.126 A matrix has eigenvalues –1 and –2. The corresponding eigen vectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. The matrix is

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$
(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$
(d) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

[EE, GATE-2013, 2 Marks]

Solution: (d)

$$AX = \lambda X$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a - b = -1 \quad \text{...(i)}$$

$$c - d = 1 \quad \text{...(ii)}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \quad a - 2b = -2 \quad \text{...(iii)}$$

$$c - 2d = 4 \quad \text{...(iv)}$$

From equation (i) and (iii), a = 0 and b = 1

From equation (ii) and (iv), c = –2 and d = –3

$$\therefore \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Q.127 A system matrix is given as follows.

$$A\begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

The absolute value of the ratio of the maximum eigenvalue to the minimum eigenvalue is ______.

[EE, GATE-2014 : 2 Marks, Set-1]

Solution :

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & -1 \\ -6 & -11-\lambda & 6 \\ -6 & -11 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda[-55 + 11\lambda - 5\lambda + \lambda^2 + 66] - 1[-30 + 6\lambda + 36] - 1[66 - 66 - \lambda 6] = 0$

$\Rightarrow -\lambda(\lambda^2 + 6\lambda + 11) - 1(6\lambda + 6) + 6\lambda = 0$

$\Rightarrow -\lambda^3 - 6\lambda^2 - 11\lambda - 6 = 0$

$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\Rightarrow (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$

$\Rightarrow \lambda = -1, -2, -3$

Maximum eigen value is -1 of λ are $|\lambda| = 1, 2, 3.$

Ratio of maximum and minimum eigen value is $= 3 : 1 = \frac{3}{1} = 3$

Q.128 The maximum value of "a" such that the matrix $\begin{pmatrix} -3 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & a & -2 \end{pmatrix}$ has three linearly independent real eigenvectors is

(a) $\frac{2}{3\sqrt{3}}$

(b) $\frac{1}{3\sqrt{3}}$

(c) $\frac{1+2\sqrt{3}}{3\sqrt{3}}$

(d) $\frac{1+\sqrt{3}}{3\sqrt{3}}$

[EE, GATE-2015 : 2 Marks, Set-1]

Solution: (b)

Characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & 0 & -2 \\ 1 & -1-\lambda & 0 \\ 0 & a & -2-\lambda \end{vmatrix} = 0$$

$$-(3+\lambda)[(1+\lambda)(2+\lambda) - 0] - 2(a - 0) = 0$$

$$2a = -(\lambda + 1)(\lambda + 2)(\lambda + 3) = -(\lambda + 1)(\lambda^2 + 5\lambda + 6)$$

$$2a = -(\lambda^3 + 6\lambda^2 + 11\lambda + 6)$$

$$\frac{2da}{d\lambda} = -(3\lambda^2 + 12\lambda + 11) = 0 \quad \text{(for a maxima and minima)}$$

$$3\lambda^2 + 12\lambda + 11 = 0$$

$$\lambda = \frac{-12 \pm \sqrt{144 - 132}}{6} = -2 \pm \frac{1}{\sqrt{3}}$$

$$\lambda = -2 + \frac{1}{\sqrt{3}}$$

$$2a = -\left(-2 + \frac{1}{\sqrt{3}} + 1\right)\left(-2 + \frac{1}{\sqrt{3}} + 2\right)\left(-2 + \frac{1}{\sqrt{3}} + 3\right)$$

$$= -\left(\frac{1}{\sqrt{3}}-1\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}+1\right) = -\left(\frac{1}{3}-1\right)\left(\frac{1}{\sqrt{3}}\right)$$

$$2a = \frac{2}{3}\times\frac{1}{\sqrt{3}}$$

$$a = \frac{1}{3\sqrt{3}}$$

Q.129 The eigenvalues and the corresponding eigenvectors of a 2 × 2 matrix are given by

Eigenvalue	Eigenvector
$\lambda_1 = 8$	$v_1 = \begin{bmatrix}1\\1\end{bmatrix}$
$\lambda_2 = 4$	$v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

The matrix is

(a) $\begin{bmatrix}6 & 2\\2 & 6\end{bmatrix}$ (b) $\begin{bmatrix}4 & 6\\6 & 4\end{bmatrix}$

(c) $\begin{bmatrix}2 & 4\\4 & 2\end{bmatrix}$ (d) $\begin{bmatrix}4 & 8\\8 & 4\end{bmatrix}$ **[EC, GATE-2006, 2 marks]**

Solution: (a)

By property of eigen values, sum of diagonal elements should be equal to sum of values of λ.

So, $\Sigma \lambda_i = \lambda_1 + \lambda_2 = 8 + 4 = 12 = \text{Trace (A)}$

Only in choice (a), Trace (A) = 12.

Q.130 All the four entries of the 2 × 2 matrix $P = \begin{bmatrix}p_{11} & p_{12}\\p_{21} & p_{22}\end{bmatrix}$ are nonzero, and one of its eigenvalues is zero. Which of the following statements is true?

(a) $p_{11}p_{22} - p_{12}p_{21} = 1$ (b) $p_{11}p_{22} - p_{12}p_{21} = -1$

(c) $p_{11}p_{22} - p_{12}p_{21} = 0$ (d) $p_{11}p_{22} + p_{12}p_{21} = 0$

[EC, GATE-2008, 1 mark]

Solution: (c)

Since, $\Pi \lambda_i = |A|$

and If one of the eigen values is zero, then

$$\Pi \lambda_i = |A| = 0$$

Now, $|A| = \begin{vmatrix}p_{11} & p_{12}\\p_{21} & p_{22}\end{vmatrix} = 0$

$\Rightarrow$ $p_{11}\,p_{22} - p_{12}\,p_{21} = 0$

Which is choice (c).

Q.131 The eigen values of the following matrix are $\begin{bmatrix}-1 & 3 & 5\\-3 & -1 & 6\\0 & 0 & 3\end{bmatrix}$

(a) 3, 3 + 5j, 6 – j (b) –6 + 5j, 3 + j, 3 – j

(c) 3 + j, 3 – j, 5 + j (d) 3, –1 + 3j, –1–3j

[EC, GATE-2009, 2 marks]

Solution: (d)

Sum of eigen values = $Tr(A) = -1 + -1 + 3 = 1$

So, $\Sigma\lambda_i = 1$

Only choice (d) $(3, -1+3j, -1-3j)$ gives $\Sigma\lambda_i = 1$.

Q.132 The eigenvalues of a skew-symmetric matrix are

(a) always zero
(b) always pure imaginary
(c) either zero or pure imaginary
(d) always real

[EC, GATE-2010, 1 mark]

Solution: (c)

Eigen values of a skew symmetric matrix are either zero or pure imaginary.

Q.133 The minimum eigen value of the following matrix is

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

(a) 0
(b) 1
(c) 2
(d) 3

[EC, GATE-2013, 1 Mark]

Solution: (a)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(12-\lambda)(5-\lambda)-49]-5[5(5-\lambda)-14]+2[35-2(12-\lambda)]=0$$

$$(3-\lambda)\left[60-17\lambda+\lambda^2-49\right]-5(25-5\lambda-14)+2(35-24+2\lambda)=0$$

$$(3-\lambda)(\lambda^2-17\lambda+11)-5(11-5\lambda)+2(11+2\lambda)=0$$

$$3\lambda^2-51\lambda+33-\lambda^3+17\lambda^2-11\lambda-55+25\lambda+22+4\lambda=0$$

$$-\lambda^3+20\lambda^2-33\lambda = 0$$

$$\lambda^3-20\lambda^2+33\lambda = 0$$

$$\lambda(\lambda^2-20\lambda+33) = 0$$

$$\lambda = 0, 1.82, 18.2$$

So minimum eigen value is 0.

Q.134 A real (4×4) matrix A satisfies the equation $A^2 = I$, where I is the (4×4) identity matrix. The positive eigen value of A is ______.

[EC, GATE-2014 : 1 Mark, Set-1]

Solution :

Since, $A^2 = I$, $\quad eig(A^2) = eig(I) = 1$

$\Rightarrow \quad eig(A)^2 = 1$

$\Rightarrow \quad eig(A) = \pm 1$

Therefore, the positive eigen value of A is +1.

Q.135 The value of x for which all the eigen-values of the matrix given below are real is

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

(a) $5 + j$
(b) $5 - j$
(c) $1 - 5j$
(d) $1 + 5j$ **[EC, GATE-2015 : 1 Mark, Set-2]**

Solution: (b)

For a matrix containing complex number, eigen values are real if and only if

$$A = A^{\theta} = (\bar{A})^T$$

$$A = \begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} 10 & \bar{x} & 4 \\ 5-j & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

By comparing these, $x = 5 - j$

Q.136 Consider the following matrix.

$$A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$$

If the eigenvalues of A are 4 and 8, then

(a) x = 4, y = 10
(b) x = 5, y = 8
(c) x = –3, y = 9
(d) x = –4, y = 10

[CS, GATE-2010, 2 marks]

Solution: (d)

$$\text{Sum of eigen values} = \text{Trace (A)} = 2 + y$$

$$\text{Product of eigen values} = |A| = 2y - 3x$$

$$\therefore \quad 4 + 8 = 2 + y \quad \text{... (i)}$$

$$4 \times 8 = 2y - 3x \quad \text{... (ii)}$$

$$\therefore \quad 2 + y = 12 \quad \text{... (i)}$$

$$2y - 3x = 32 \quad \text{... (ii)}$$

$\therefore$ Solving (i) and (ii) we get x = –4 and y = 10.

Q.137 Consider the matrix as given below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

Which one of the following options provides the CORRECT values of the eigenvalues of the matrix?

(a) 1, 4, 3
(b) 3, 7, 3
(c) 7, 3, 2
(d) 1, 2, 3

[CS, GATE-2011, 2 marks]

Solution: (a)

Since the given matrix is upper triangular, its eigen values are the diagonal elements themselves, which are 1, 4 and 3.

Q.138 The value of the dot product of the eigenvectors corresponding to any pair of different eigen values of a 4-*by*-4 symmetric positive definite matrix is ______.

[CS, GATE-2014 : 1 Mark, Set-1]

Solution :

The value of the dot product of the eigenvectors corresponding to any pair of different eigen values of a 4-*by*-4 symmetric positive definite matrix is 0.

Q.139 The product of the non-zero eigenvalues of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is ______.

[CS, GATE-2014 : 2 Marks, Set-2]

Solution :

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ let } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$Ak = Xk$

$\Rightarrow \quad x_1 + x_5 = kx_1 = kx_5$

$\Rightarrow \quad x_2 + x_3 + x_4 = kx_2 = kx_4$

(i) $k \neq 0$

say, $x_1 = x_5 = a$

$x_2 = x_3 = x_4 = b$

$\Rightarrow \quad x_1 + x_5 = kx_1$

$\Rightarrow \quad 2a = ka$

$\Rightarrow \quad k = 2$

$\Rightarrow \quad x_2 + x_3 + x_4 = kx_2$

$\Rightarrow \quad 3b = kb$

$\Rightarrow \quad k = 3$

(ii) $k = 0$

$\Rightarrow$ Eigen value $k = 0$

$\therefore$ There are 3 distinct eigen values: 0,2,3

Product of non-zero eigen values: $2 \times 3 = 6$

Q.140 Which one of the following statements is TRUE about every $n \times n$ matrix with only real eigenvalues?

(a) If the trace of the matrix is positive and the determinant of the matrix is negative, at least one of its eigenvalues is negative.

(b) If the trace of the matrix is positive, all its eigenvalues are positive.

(c) If the determinant of the matrix is positive, all its eigenvalues are positive.

(d) If the product of the trace and determinant of the matrix is positive, all its eigenvalues are positive.

[CS, GATE-2014 : 1 Mark, Set-3]

Solution : (a)

If either the trace or determinant is positive, there exist at least one positive eigen value.

Trace of the matrix is positive and the determinant of the matrix is negative, this is possible only when there is odd number of negative eigen values. Hence at least one eigen value is negative.

Q.141 Consider the following 2 × 2 matrix A where two elements are unknown and are marked by a and b. The eigenvalues of this matrix are −1 and 7. What are the values of a and b?

$$A = \begin{pmatrix} 1 & 4 \\ b & a \end{pmatrix}$$

(a) a = 6, b = 4 (b) a = 4, b = 6

(c) a = 3, b = 5 (d) a = 5, b = 3

[CS, GATE-2015 : 2 Marks, Set-1]

Solution: (d)

$$\text{Trace} = \text{Sum of eigen values}$$

$$1 + a = 6$$

$$\Rightarrow \quad \mathbf{a = 5}$$

Determinant = Product of eigen values

$$(a - 4b) = -7$$

$$5 - 4b = -7$$

$$-4b = -12$$

$$\Rightarrow \quad \mathbf{b = 3}$$

$$\therefore \quad a = 5, b = 3$$

Q.142 Consider a linear time invariant system $\dot{x} = Ax$, with initial conditions $x(0)$ at $t = 0$. Suppose a and b are eigenvectors of (2 × 2) matrix A corresponding to distinct eigenvalues λ_1 and λ_2 respectively. Then the response $x(t)$ of the system due to initial condition $x(0) = \alpha$ is

(a) $\alpha e^{\lambda_1 t}$ (b) $e^{\lambda_2 t}\beta$

(c) $e^{\lambda_2 t}\alpha$ (d) $e^{\lambda_1 t}\alpha + e^{\lambda_2 t}\beta$

[EE, 2016 : 2 Marks, Set-2]

Solution: (a)

$$\dot{x} = Ax$$

Eigen values are λ_1 and λ_2

We can write,

$$\phi(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Response due to initial conditions,

$$x(t) = \phi(t) \cdot x(0)$$

$$x(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha\, e^{\lambda_1 t}$$

Q.143 Let the eigenvalues of a 2 × 2 matrix A be 1, −2 with eigenvectors x_1 and x_2 respectively. Then the eigenvalues and eigenvectors of the matrix $A^2 - 3A + 4I$ would, respectively, be

(a) 2, 14; x_1, x_2

(b) 2, 14; $x_1 + x_2, x_1 - x_2$

(c) 2, 0; x_1, x_2

(d) 2, 0; $x_1 + x_2, x_1 - x_2$

[EE, 2016 : 2 Marks, Set-1]

Solution: (a)

Eigen values of $A^2 - 3A + 4I$ are

$$= (1)^2 - 3(1) + 4 \text{ and } (-2)^2 - 3(-2) + 4$$
$$= 2, 14$$

Note: $A^2X = l^2X$

$\Rightarrow$ X is eigen vector for A^2 corresponding to eighen value l^2

X_1 and X_2 are e v of A corresponding to 1, −2

Then X_1 and X_2 are $e.v$ of $A^2 - 3 + 4I$ corresponding to 2, 14.

Q.144 The number of linearly independent eigenvectors of matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is ________

[ME, 2016 : 2 Marks, Set-3]

Solution:

Consider $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

Ch. equation is $|A - \lambda I| = 0$

$(2-\lambda)(2-\lambda)(3-\lambda) = 0$

$$\lambda = 2, 2, 3$$

$\lambda = 3$ there is one L.I. Eigen vector

$\lambda = 2$ Consider $(A - 2I)\,x = 0$

rank = 2 The equation are $x_2 = 0$

No. of variables = 3 $x_3 = 0$

Let $x_1 = k$ be independent.

$\therefore$ Eigen vector is $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Only one independent Eigen vector in the case of $\lambda = 2$

Hence finally no. of L.I. Eigen vectors = 2

Q.145 The value of x for which the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$$ has zero as an eigen value is ____.

[EC, 2016 : 1 Mark, Set-2]

Solution:

A has an eigen value is zero

$\therefore$ $|A| = 0$

$$\begin{vmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{vmatrix} = 0$$

$3(-63 + 7x + 52) - 2(-81 + 9x + 78) + 4(-36 + 42) = 0$

$3(7x - 11) - 2(9x - 3) + 4(6) = 0$

$21x - 33 - 18x + 6 + 24 = 0$

$3x - 3 = 0$

$x = 1$

Q.146 Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$ whose eigenvalues are 1, –1 and 3. Then Trace of $(A^3 - 3A^2)$ is____.

[IN, 2016 : 2 Marks]

Solution

Eigne values of given matrix A are 1 –1, 3

Eigen values of A^3 are	1,	–1,	27
Eigen values of $3A^2$ are	3,	3,	27
Eigen values of $A^3 - 3A^2$ are	–2,	–4,	0

trace of $A^3 - 3A^2 = -2 - 4 + 0 = -6$

Q.147 Suppose that the eigenvalues of matrix A are 1, 2, 4. The determinant of $(A^{-1})^T$ is______.

[CS, 2016 : 1 Mark, Set-2]

Solution:

Eigen(A) = 1, 2, 4 $\Rightarrow |A| = 1\times2\times4 = 8$

Now, $|(A^{-1})^t| = |A^{-1}| = \frac{1}{|A|} = \frac{1}{8} = 0.125$

Q.148 Suppose that the eigenvalues of matrix A are 1, 2, 4. The determinant of $(A^{-1})^T$ is ______

[CS, 2016 : 1 Mark, Set-2]

Solution:

$$\text{Eigen}(A) = 1, 2, 4 \Rightarrow |A| = 1\times 2\times 4 = 8$$

$$\text{Now, } |(A^{-1})^T| = |A^{-1}| = \frac{1}{|A|} = \frac{1}{8} = 0.125$$

1.8.5 The Cayley-Hamilton Theorem

This theorem is an interesting one that provides an alternative method for finding the inverse of a matrix A. Also any positive integral power of A can be expressed, using this theorem, as a linear combination of those of lower degree. We give below the statement of the theorem without proof.

Statement of the Theorem: Every square matrix satisfies its own characteristic equation.

This means that, if $c_0 \lambda^n + c_1 \lambda^{n-1} + \dots + c_{n-1} \lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$$c_0 A^n + c_1 A^{n-1} + \dots + c_{n-1} A + c_n I = 0 \quad \dots (i)$$

Note: when λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by $c_n I$ to get the result of Cayley-Hamilton theorem, where I is the unit matrix of order n.

Also 0 in the R.H.S. of (i) is a null matrix of order n.

1.8.5.1 Finding Inverse off a Matrix by using Cayley-Hamilton Theorem

Example: Find A^{-1} by Cayley-Hamilton theorem, if

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

By Cayley-Hamilton theorem

$$A^2 - 3A - 10 I = 0$$

$$\Rightarrow I = \frac{1}{10}\left[A^2 - 3A\right]$$

Pre-multiplying by A^{-1} we get

$$A^{-1} = \frac{1}{10}[A - 3I] = \frac{1}{10}\left(\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{10}\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{10} \\ \frac{2}{5} & -\frac{1}{10} \end{bmatrix}$$

1.8.5.2 Finding Higher Powers of a Matrix in Terms of its Lower Powers

Example: If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, express A^5 as a linear polynomial in A.

characteristic equation is $\lambda^2 - 3\lambda - 10 = 0$

by Cayley-Hamilton theorem,

$$A^2 - 3A - 10\,I = 0$$

$$\Rightarrow \quad A^2 = 3A + 10\,I$$

If A is n × n matrix, any power of A can be written as a polynomial of maximum degree n – 1. Here, since A is 2 × 2, we can write any power of A as a polynomial of degree 1, i.e., a linear polynomial of A, as shown below.

$$A^2 = 3A + 10I \quad \text{... (i)}$$

$$A^3 = 3A^2 + 10A \quad \text{... (ii)}$$

substituting (i), again in (ii), we get

$$A^3 = (3A + 10I) + 10A = 19A + 30\,I \quad \text{... (iii)}$$

Now

$$A^4 = 19\,A^2 + 30\,A$$

again we substitute equation (i) in equation (iii) to get,

$$A^4 = 19(A + 10I) + 30A = 87\,A + 190\,I \quad \text{... (iv)}$$

Now

$$A^5 = 87\,A^2 + 190\,A$$

again substituting equation (i) in equation (iv) we get,

$$A^5 = 87\,(3A + 10I) + 190\,A = 451\,A + 870\,I$$

Which is the desired result.

1.8.5.3 Expressing Any Matrix Polynomial in A of size n × n as a Polynomial of Degree n –1 in A by using Cayley-Hamilton Theorem

Example: Process to express a polynomial of a 2 × 2 Matrix as a linear polynomial in A:

Example: Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ Express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Step 1: First of all write the characteristic equation of A.
In this case.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) + 1$$

$$= \lambda^2 - 5\lambda + 7$$

Thus the characteristic equation of A is $|A - \lambda I| = 0$

i.e., is $\lambda^2 - 5\lambda + 7 = 0$ ... (i)

Step 2: By Cayley Hamilton theorem, matrix A satisfies the equation (i). Therefore, putting A = I in (i) we get

$$A^2 - 5A + 7 = 0$$

$$\Rightarrow \quad A^2 = 5A - 7I \quad \text{... (ii)}$$

Step 3: Find the A^5, A^4, A^3 with the help of (ii). In this case

$$A^3 = 5A^2 - 7A$$

$$\Rightarrow \quad A^4 = 5A^3 - 7A^2$$

$$\Rightarrow \quad A^4 = 5A^4 - 7A^3$$

$$2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$$

$$= 7A^4 - 14A^3 + A^2 - 4I = 7[5A^3 - 7A^2] - 14A^3 + A^2 - 4I$$

$$= 21A^3 - 48A^2 - 4I = 21(5A^2 - 7A) - 48A^2 - 4I$$

$$= 57A^2 - 147A - 4I = 57[5A - 7I] - 147A - 4I = 138A - 403I$$

⇒ which is a linear polynomial in A.

ILLUSTRATIVE EXAMPLES FROM GATE

Statement for Linked Answer Question 149 and 150.

Cayley-Hamilton Theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

Q.149 A satisfies the relation

(a) $A + 3I + 2A^{-1} = 0$ (b) $A^2 + 2A + 2I = 0$

(c) $(A + I)(A + 2I) = I$ (d) $\exp(A) = 0$

[EE, GATE-2007, 2 marks]

Solution: (a)

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(-\lambda) + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

A will satisfy this equation according to Cayley-Hamilton theorem

i.e. $A^2 + 3A + 2I = 0$

multiplying by A^{-1} on both sides we get

$$A^{-1}A^2 + 3A^{-1}A + 2A^{-1}I = 0$$

$$A + 3I + 2A^{-1} = 0$$

Q.150 A^9 equals

(a) $511A + 510I$ (b) $309A + 104I$

(c) $154A + 155I$ (d) $\exp(9A)$

[EE, GATE-2007, 2 marks]

Solution: (a)

To calculate A^9

start from $A^2 + 3A + 2I = 0$ which has been derived above

$\Rightarrow$

$$A^2 = -3A - 2I$$

$$A^4 = A^2 \times A^2 = (-3A - 2I)(-3A - 2I) = 9A^2 + 12A + 4I$$

$$= 9(-3A - 2I) + 12A + 4I = -15A - 14I$$

$$A^8 = A^4 \times A^4 = (-15A - 14I)(-15A - 14I)$$

$$= 225A^2 + 420A + 156I = 225(-3A - 2I) + 420A \;\; 196I$$

$$= -255A - 254I$$

$$A^9 = A \times A^8$$

$$= A(-255A - 254I)$$

$$= -255A^2 - 254A$$

$$= -255(-3A - 2I) - 254A$$

$$= 511A + 510I$$

Q.151 The characteristic equation of a (3 × 3) matrix P is defined as

$a(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$

If I denotes identity matrix, then the inverse of matrix P will be

(a) $(P^2 + P + 2I)$

(b) $(P^2 + P + 1)$

(c) $-(P^2 + P + 1)$

(d) $-(P^2 + P + 2I)$

[EE, GATE-2008, 1 mark]

Solution: (d)

If characteristic equation is

$$\lambda^3 + \lambda^2 + 2\lambda + 1 = 0$$

Then by cayley - hamilton theorem,

$$P^3 + P^2 + 2P + I = 0$$

$$I = -P^3 - P^2 - 2P$$

Multiplying by P^{-1} on both sides,

$$P^{-1} = -P^2 - P - 2I = -(P^2 + P + 2I)$$

Q.152 Given that

$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

the value A^3 is

(a) 15 A + 12 I

(b) 19 A + 30 I

(c) 17 A + 15 I

(d) 17 A + 21 I

[EC, EE, IN GATE-2012, 2 marks]

Solution: (b)

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix} -5-\lambda & -3 \\ 2 & 0-\lambda \end{vmatrix} = 0$$

$$(-5 - \lambda)(-\lambda) + 6 = 0$$

$\Rightarrow$ $$\lambda^2 + 5\lambda + 6 = 0$$

So, $$A^2 + 5A + 6I = 0 \quad \text{(by Cayley Hamilton theorem)}$$

$\Rightarrow$ $$A^2 = -5A - 6I$$

Multiplying by A on both sides, we have,

$$A^3 = -5A^2 - 6A$$

$\Rightarrow$ $$A^3 = -5(-5A - 6I) - 6A$$

$$= 19A + 30I$$

Q.153 A 3 × 3 matrix P is such that, $P^3 = P$. Then the eigenvalues of P are

(a) 1, 1, –1

(b) 1, 0.5 + j0.866, 0.5 – j0.866

(c) 1, –0.5 + j0.866, – 0.5 – j0.866

(d) 0, 1, –1

[EE, 2016 : 1 Mark, Set-2]

Solution: (d)

By Calyey Hamilton theorem,

$$\lambda^3 = \lambda$$
$$\lambda = 0, 1, -1$$

Q.154 A sequence $x[n]$ is specified as

$$\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ for } n \geq 2.$$

The initial conditions are $x[0] = 1$, $x[1] = 1$, and $x[n] = 0$ for $n < 0$. The value of $x[12]$ is ______

[EC, 2016 : 2 Marks, Set-1]

Solution:

For $$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

equation $$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$-\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - \lambda - 1 = 0$$

By Cayley Hamilton Theorem

$$A^2 - A - I = 0$$
$$A^2 = A + I$$
$$A^4 = A^2 + 2A + I = A + I + 2A + I = 3A + 2I$$
$$A^8 = 9A^2 + 12A + 4I = 9(A + I) + 12A + 4I = 21\,A + 13I$$
$$A^{12} = A^4 \cdot A^8 = 144A + 89\,I = \begin{bmatrix} 233 & 144 \\ 144 & 89 \end{bmatrix}$$
$$\begin{bmatrix} x[12] \\ x[11] \end{bmatrix} = \begin{bmatrix} 233 & 144 \\ 144 & 89 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$x[12] = 233$$

1.8.6 Similar Matrices

Two matrices A and B are said to be similar, if there exists a non-singular matrix P such that $B = P^{-1}AP$.

1.8.6.1 Properties of Similar Matrices

1. A is always similar to A.

   Proof: Since $A = I^{-1}AI$ and I is always non-singular, therefore A is similar to A.

2. If A is similar to B then B is also similar to A.

   Proof: If A is similar to B then $B = P^{-1}AP$ (where P is non-singular)

   Premultiplying above equation by P and postmultiplying by P^{-1}, we get $PBP^{-1} = PP^{-1}APP^{-1} = A$.

   i.e., $A = PBP^{-1}$

   So B is also similar to A.

3. If A is similar to B and B is similar to C then A is similar to C.

   Proof: A is similar to B $\Rightarrow B = P^{-1}AP$ ...(i)

   B is similar to C $\Rightarrow C = Q^{-1}BQ$ ...(ii)

Substituting eq. (i) and (ii) we get

$C = Q^{-1}P^{-1}APQ$

Now putting PQ = D, we get $C = D^{-1}AD$, which proves that A is similar to C.

4. Combining properties 1, 2 and 3 above we can say that the similarity relation between matrices is reflexive, symmetric and transitive and hence an equivalence relation.
5. Similar matrices have the same eigenvalues.

1.8.7 Diagonalisation of a Matrix

Finding the a matrix D which is a diagonal matrix and which is similar to A is called diagonalisation. i.e., we wish to find a non-singular matrix M such that

$$A = M^{-1}DM$$

where D is a diagonal matrix.

Condition for a Matrix to be Diagonalisable:

1. A necessary and sufficient condition for a matrix $A_{n \times n}$ to be diagonalisable is that the matrix must have n linearly independent eigen vectors.
2. A sufficient (but not necessary) condition for a matrix $A_{n \times n}$ to be diagonalisable is that the matrix must have n linearly independent eigen values.

   This is because if a matrix has n linearly independent eigen values then it surely has n linearly independent eigen vectors (although the converse of this is not true).

When A is diagonalisable $A = M^{-1}DM$, where the matrix D is a diagonal matrix constructed using the eigen values of A as its diagonal elements. Also the corresponding matrix M can be obtained by constructing a n × n matrix whose columns are the eigen vectors of A.

Practical application of Diagonalisation:

One of the uses of diagonalisation is for computing higher powers of a matrix efficiently.

If $A = M^{-1}DM$ then $A^n = M^{-1}D^n M$

The above property makes it easy to compute higher powers of a matrix A, since computing D^n is much more easy compared with computing A^n.

OOOO

CHAPTER 2

Calculus

2.1 LIMIT

2.1.1 Definition

A number A is said to be limit of a function f(x) at $x = a$ iff for any arbitrarily choosen positive integer ϵ, however small but not zero there exist a corresponding number δ greater than zero such that: $|f(x) - A| < \epsilon$ for all values of x for which $0 < |x - a| < \delta$ where $|x - a|$ means the absolute value of $(x - a)$ without any regard to sign.

2.1.2 Right and Left Hand Limits

If x approaches a from the right, that is, from larger value of x than a, the limit of f as defined before is called the right hand limit of f(x) and is written as:

$$\underset{x \to a+0}{\text{Lt}} f(x) \text{ or } f(a+0) \text{ or } \underset{x \to a^+}{\text{Lt}} f(x)$$

Working rule for finding right hand limit is, put a + h for x in f(x) and make h approach zero.

In short, we have, $$f(a+0) = \underset{h \to 0}{\text{Lt}} f(a+h)$$

Similarly if x approaches a from left, that is from smaller values of x than a, the limit of f is called the left hand limit and is written as:

$$\underset{x \to a-0}{\text{Lt}} f(x) \text{ or } f(a-0) \text{ or } \underset{x \to a^-}{\text{Lt}} f(x)$$

In this case, we have, $$f(a-0) = \underset{h \to 0}{\text{Lt}} f(a-h)$$

If both right hand and left hand limit of f, as $x \to a$ exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If however, either or both of these limits do not exist, the limit of f as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

$\therefore$ when $$\underset{x \to a^+}{\text{Lt}} f(x) = \underset{x \to a^-}{\text{Lt}} f(x)$$

then $$\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a^+}{\text{Lt}} f(x) = \underset{x \to a^-}{\text{Lt}} f(x)$$

Limit of a function can be any real number, ∞ or $-\infty$. It an sometimes be ∞ or $-\infty$, which are also allowed values for limit of a function.

2.1.3 Various Formulae

These formulae are sometimes useful while taking limits.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots..$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots..$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots...$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots...$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.. \; |x|<1$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots..\right) |x|<1$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots..$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots..$$

$$\sin h\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots..$$

$$\cos h\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots..$$

Remember: $\log 1 = 0$; $\log e = 1$; $\log \infty = \infty$; $\log 0 = -\infty$

2.1.4 Some Useful Results

1. $\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = 1$
2. $\underset{x\to 0}{\text{Lt}} \cos x = 1$
3. $\underset{x\to 0}{\text{Lt}} \frac{\tan x}{x} = 1$
4. $\underset{x\to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e$
5. $\underset{x\to 0}{\text{Lt}} (1+nx)^{\frac{1}{x}} = e^n$
6. $\underset{x\to \infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$
7. $\underset{x\to \infty}{\text{Lt}} \left(1+\frac{a}{x}\right)^x = e^a$

2.1.5 Indeterminate Forms

A fraction whose numerator and denominator both tend to zero as $x \to a$ is an example of an indeterminate form written as 0/0. It has no definite values. Other indeterminate forms are: ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^0.

(Indeterminate form are not any definite number and hence are not acceptable as limits. To find limit in such cases, we use the L'hospital's rule)

2.1.5.1 Indeterminate Form-I $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

Use L'hospital's Rule.

L'Hospital Rule: If f(x) and $\phi(x)$ be two functions of x and if,

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} \phi(x) = 0$$

or if
$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} \phi(x) = \infty,$$

then
$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)}$$

provided, the latter limit exists, finite or infinite.

Working Rule: If the limit of $f(x)/\phi(x)$ as $x \to a$ takes the form 0/0, differentiate the numerator and denominator separately with respect to x and obtain a new function $f'(x)/\phi'(x)$. Now as $x \to a$ if it again takes the form 0/0, differentiate the numerator and denominator again with respect to x and repeat the above process, until the indeterminate form is removed and we get either a real number, ∞ or $-\infty$ as a limit.

Caution: Before applying L'Hospital's rule at any stage, be sure that the form is 0/0. Do not go on applying this rule, if the form is not 0/0.

2.1.5.2 Indeterminate Form-II ($0 \times \infty$)

This form can be easily reduced to the form 0/0 or to the form ∞/∞, and then L'Hospital's rule may be applied.

$$\text{Let } \underset{x \to a}{\text{Limit}}\, f(x) = 0 \text{ and } \underset{x \to a}{\text{Limit}}\, \phi(x) = \infty.$$

Then we can write

$$\underset{x \to a}{\text{Limit}}\, f(x) \cdot \phi(x) = \underset{x \to a}{\text{Limit}} \frac{f(x)}{1/\phi(x)} \text{ [form 0/0] or } \underset{x \to a}{\text{Limit}} \frac{\phi(x)}{1/f(x)} \text{ [form } \infty/\infty]$$

Thus $\underset{x \to a}{\text{Limit}}\, f(x) \cdot \phi(x)$ is reduced to the form 0/0 or ∞/∞ which can now be evaluated by L' Hospital rule

2.1.5.3 Indeterminate Form-III (0^0 or 1^∞ or ∞^0)

Suppose $\underset{x \to a}{\text{Limit}}\, [f(x)]^{\phi(x)}$ takes any one of these three forms.

Then
$$\text{let } y = \underset{x \to a}{\text{Limit}}\, [f(x)]^{\phi(x)}$$

Taking log on both sides, we get

$$\log y = \underset{x \to a}{\text{Limit}}\, \phi(x) \cdot \log f(x)$$

Now in any of these above cases log y takes the form $0 \times \infty$ which is changed to the form 0/0 or ∞/∞ then it can be evaluated by previous methods.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 The value of the function $f(x) = \lim_{x \to 0} \frac{x^3 + x^2}{2x^3 - 7x^2}$ is

(a) 0
(b) $-\frac{1}{7}$
(c) $\frac{1}{7}$
(d) ∞

[CE, GATE-2004, 1 mark]

Solution: (b)

$$f(x) = \lim_{x \to 0} \left[\frac{x^3 + x^2}{2x^3 - 7x^2}\right]$$

Since this has $\frac{0}{0}$ form, limit can be found by repeated application of L' Hospitals rule.

$$f(x) = \lim_{x \to 0} \left[\frac{3x^2 + 2x}{6x^2 - 14x}\right] = \lim_{x \to 0} \left[\frac{6x + 2}{12x - 14}\right] = \left[\frac{6 \times 0 + 2}{12 \times 0 - 14}\right] = -\frac{1}{7}$$

Q.2 The $\lim_{x \to 0} \frac{\sin\left[\frac{2}{3}x\right]}{x}$ is

(a) 2/3
(b) 1
(c) 3/2
(d) ∞

[CE, GATE-2010, 1 mark]

Solution: (a)

$$\lim_{x \to 0} \frac{\sin\left(\frac{2}{3}x\right)}{x} = \lim_{\frac{2}{3}x \to 0} \frac{\sin\left(\frac{2}{3}x\right)}{\frac{2}{3}x} \cdot \frac{2}{3} = (1)\left(\frac{2}{3}\right) = \frac{2}{3}$$

Q.3 $\lim_{x \to \infty}\left(\frac{x + \sin x}{x}\right)$ equal to

(a) $-\infty$
(b) 0
(c) 1
(d) ∞

[CE, GATE-2014 : 1 Mark, Set-1]

Solution : (c)

$$\lim_{x \to \infty}\left(\frac{x + \sin x}{x}\right) = \frac{1 + \frac{\sin x}{x}}{1} = \frac{1 + 0}{1} = 1$$

Since, $$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

Q.4 The expression $\lim_{\alpha \to 0} \frac{x^{\alpha} - 1}{\alpha}$ is equal to

(a) $\log x$
(b) 0
(c) $x \log x$
(d) ∞

[CE, GATE-2014 : 2 Marks, Set-2]

Solution : (a)

$$\lim_{\alpha \to 0} \frac{x^{\alpha}-1}{\alpha} \left[\frac{0}{0}\text{ form}\right]$$

Use *L*-Hospital Rule (Note: Differentiate numerator and denominator w.r.t. α keeping x as constant.)

$$\Rightarrow \quad \lim_{\alpha \to 0} \frac{x^{\alpha} \ln x}{1} = \log x$$

Q.5 $\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to

(a) e^{-2}
(b) e
(c) 1
(d) e^2 **[CE, GATE-2015 : 1 Mark, Set-II]**

Solution: (d)

$$y = \lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{2x}$$

$$\Rightarrow \quad \log y = \lim_{x \to \infty} 2x \log\left(1+\frac{1}{x}\right)$$

Which is in the form of $\infty \times 0$.

To convert this into $\frac{0}{0}$ form, we rewrite as

$$\Rightarrow \quad \log y = \lim_{x \to \infty} \frac{2\log\left(1+\frac{1}{x}\right)}{1/x}$$

Now it is in $\frac{0}{0}$ form.

Using L'Hospital's rule

$$\Rightarrow \quad \log y = \lim_{x \to \infty} \frac{\dfrac{2 \times -\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{2}{1+\frac{1}{x}} = 2$$

$$\therefore \quad y = e^2$$

Q.6 $\underset{x \to 0}{\text{Lt}} \frac{\sin^2 x}{x}$ is equal to

(a) 0
(b) ∞
(c) 1
(d) -1 **[ME, GATE-2003, 1 mark]**

Solution: (a)

$$\underset{x \to 0}{\text{Lt}} \frac{\sin^2 x}{x} = \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^2 x = 1 \times 0 = 0$$

Q.7 If $f(x) = \frac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$, then $\lim_{x \to 3} f(x)$ will be

(a) − 1/3
(b) 5/18
(c) 0
(d) 2/5

[ME, GATE-2006, 2 marks]

Solution: (b)

$$\underset{x \to 3}{\text{Lt}}\, f(x) = \underset{x \to 3}{\text{Lt}} \left(\frac{2x^2 - 7x + 3}{5x^2 - 12x - 9} \right)$$

Here this is of the form of $\left(\frac{0}{0}\right)$

So, applying Lee Hospital's rule

$$\underset{x \to 3}{\text{Lt}} \left(\frac{4x - 7}{10x - 12} \right) = \frac{5}{18}$$

Q.8 $\lim_{x \to 0} \frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3} =$

(a) 0
(b) 1/6
(c) 1/3
(d) 1

[ME, GATE-2007, 2 marks]

Solution: (b)

$$\underset{x \to 0}{\text{Lt}} \frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3}$$

This is of the form of $\left(\frac{0}{0}\right)$

Applying L' Hospital rule $\underset{x \to 0}{\text{Lt}} \frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3} \left(\frac{0}{0}\right)$

$$\underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{6x}$$

$$\underset{x \to 0}{\text{Lt}} \frac{e^x}{6} = \frac{1}{6}$$

Q.9 The Value of $\lim_{x \to 8} \frac{x^{1/3} - 2}{(x - 8)}$

(a) $\frac{1}{16}$
(b) $\frac{1}{12}$
(c) $\frac{1}{8}$
(d) $\frac{1}{4}$

[ME, GATE-2008, 1 mark]

Solution: (b)

$$(x - 8) = h \text{ (say)}$$
$$\Rightarrow \quad x = 8 + h$$

$$\therefore \quad \lim_{h \to 0} \frac{(8+h)^{1/3} - 2}{h}$$

Above form in the $\left(\frac{0}{0}\right)$ by putting the value h = 0

Applying L' hospital rule

$$\lim_{h \to 0} \frac{\frac{1}{3}(8+h)^{\left(\frac{1}{3}-1\right)}}{1} = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{12}$$

Q.10 What is $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ equal to?

(a) θ
(b) $\sin\theta$
(c) 0
(d) 1

Solution: (d) **[ME, GATE-2011, 1 mark]**

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Q.11 $\lim_{x \to 0}\left(\frac{1-\cos x}{x^2}\right)$ is

(a) 1/4
(b) 1/2
(c) 1
(d) 2

Solution: (b) **[ME, GATE-2012, 1 mark]**

$$\lim_{x \to 0} \frac{(1-\cos x)}{x^2} = \frac{1-\cos 0}{0^2} = \frac{0}{0}$$

So use L'hospitals rule

$$\lim_{x \to 0} \frac{(1-\cos x)}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{0}{0}$$

So use L'hospitals rule again

$$= \lim_{x \to 0}\left[\frac{\cos x}{2}\right] = \frac{1}{2}$$

Q.12 $\lim_{x \to 0} \frac{x - \sin x}{1 - \cos x}$

(a) 0
(b) 1
(c) 3
(d) not defined

Solution : (a) **[ME, GATE-2014 : 1 Mark, Set-1]**

$$\operatorname{Lt}_{x \to 0} \frac{x - \sin x}{1 - \cos x}$$

Applying L' Hospital's rule

$$\underset{x\to 0}{\text{Lt}} \frac{\frac{d}{dx}(x-\sin x)}{\frac{d}{dx}(1-\cos x)} = \underset{x\to 0}{\text{Lt}} \frac{(1-\cos x)}{\sin x}$$

(It is still of $\frac{0}{0}$ form)

Again applying L' Hospital's rule

$$\underset{x\to 0}{\text{Lt}} \frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(\sin x)} = \underset{x\to 0}{\text{Lt}} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

Q.13 $\underset{x\to 0}{\text{Lim}}\left(\frac{e^{2x}-1}{\sin(4x)}\right)$ is equal to

(a) 0
(b) 0.5
(c) 1
(d) 2

[ME, GATE-2014 : 1 Mark, Set-2]

Solution : (b)

$\underset{x\to 0}{\text{Lt}} \frac{(e^{2x}-1)}{\sin 4x}$, it is of $\left(\frac{0}{0}\right)$ from

Applying L' Hospital's rule,

$$\underset{x\to 0}{\text{Lt}} \frac{2e^{2x}}{4\cos 4x} = \frac{2\times 1}{4\times 1} = \frac{1}{2}$$

Q.14 The value of $\lim_{x\to 0} \frac{1-\cos(x^2)}{2x^4}$ is

(a) 0
(b) $\frac{1}{2}$
(c) $\frac{1}{4}$
(d) unefined

[ME, GATE-2015 : 1 Mark, Set-1]

Solution: (c)

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{2x^4}$$

putting the $x \to 0$

we get $\frac{0}{0}$ form

Applying L' Hospital rule

$$\Rightarrow \lim_{x\to 0} \frac{2x\sin(x^2)}{8x^3}$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin(x^2)}{4x^2}$$

$$\Rightarrow \frac{1}{4}\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$$

$$\Rightarrow \frac{1}{4}\lim_{x^2\to 0} \frac{\sin(x^2)}{x^2} = \frac{1}{4}\times 1 = \frac{1}{4}$$

Q.15 The value of $\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x+\cos x}\right)$ is ______.

[ME, GATE-2015 : 1 Mark, Set-3]

Solution: (0)

$$\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x+\cos x}\right) = \lim_{x\to 0}\left(\frac{-\sin 0}{2\sin 0+\cos 0}\right) = \frac{0}{1} = 0$$

(**Note:** Since the function is not evaluating to 0/0 not need to use L'hospital's rule)

Q.16 $\lim_{\theta\to 0}\frac{\sin(\theta/2)}{\theta}$ is

(a) 0.5
(b) 1
(c) 2
(d) not defined

[EC, GATE-2007, 1 mark]

Solution: (a)

$$\lim_{\theta\to 0}\frac{\frac{1}{2}\times\sin\left(\frac{\theta}{2}\right)}{\theta\times\frac{1}{2}} = \frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta/2}{\theta/2} = \frac{1}{2} = 0.5$$

Q.17 The value of $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ is

(a) $ln2$
(b) 1.0
(c) e
(d) ∞

[EC, GATE-2014 : 1 Mark, Set-2]

Solution : (c)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty}\frac{1}{x}\cdot x} = e^1 = e$$

Q.18 $\lim_{x\to\infty}\frac{x-\sin x}{x+\cos x}$ equals

(a) 1
(b) −1
(c) ∞
(d) $-\infty$

[CS, GATE-2008, 1 mark]

Solution: (a)

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\cos x} = \lim_{x\to\infty}\frac{1-\sin x/x}{1+\cos x/x} = \frac{\lim_{x\to\infty}(1-\sin x/x)}{\lim_{x\to\infty}(1+\cos x/x)} = \frac{1-\lim_{x\to\infty}\frac{\sin x}{x}}{1+\lim_{x\to\infty}\frac{\cos x}{x}} = \frac{1-0}{1+0} = 1$$

Q.19 What is the value of $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$?

(a) 0
(b) e^{-2}
(c) $e^{-1/2}$
(d) 1

[CS, GATE-2010, 1 mark]

Solution: (b)

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{n}\right]^{2} = \left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}\right]^{-2}$$
$$= e^{-2}$$

Q.20 $\lim\limits_{x\to\infty} x^{1/x}$ is

(a) ∞
(b) 0
(c) 1
(d) Not defined

[CS, GATE-2015 : 1 Mark, Set-1]

Solution: (c)

$$y = \lim_{x\to\infty} x^{1/x}$$
$$\log y = \lim_{x\to\infty} \log x^{1/x}$$
$$\log y = \lim_{x\to\infty} \frac{\log x}{x}$$

∞/∞ form, use L' Hospital's rule

$$\log y = \lim_{x\to\infty} \frac{1/x}{1}$$
$$\log y = 0 \quad \Rightarrow y = 1$$

Q.21 $\lim\limits_{x\to 4}\frac{\sin(x-4)}{x-4} =$ _____.

[CS, 2016 : 1 Mark, Set-1]

Solution:

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$$

Let $x - 4 = t$ not as $x \to 4$

So the requires limit is $\lim\limits_{t\to 0}\frac{\sin(t)}{t} = 1$

Q.22 The value of $\lim\limits_{x\to\infty}(1+x^2)^{e^{-x}}$ is

(a) 0
(b) 1/2
(c) 1
(d) ∞

[CS, GATE-2015 : 1 Mark, Set-3]

Solution: (c)

$\lim_{x\to\infty}(1+x^2)^{e^{-x}}$

$$\log y = \lim_{x\to\infty}\log(1+x^2)^{e^{-x}} = \lim_{x\to\infty}\frac{\log(1+x^2)}{e^x}$$

∞/∞ form apply L' Hospital's rule

$$\Rightarrow \quad \log y = \lim_{x\to\infty}\frac{\frac{1}{1+x^2}(2x)}{e^x}$$

$$\Rightarrow \quad \log y = \lim_{x\to\infty}\frac{2x}{(1+x^2)e^x}$$

Again we are getting ∞/∞ form apply L' Hospital's rule

$$\log y = \lim_{x\to\infty}\frac{2}{(1+x^2)e^x + e^x\cdot 2x}$$

$$\log y = \frac{2}{\infty} = 0$$

$$\Rightarrow \quad y = 1$$

Q.23 $\lim_{x\to 4}\frac{\sin(x-4)}{x-4} =$ _____.

[CS, 2016 : 1 Mark, Set-1]

Solution:

$\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$

Let $x - 4 = t$ not as $x \to 4$

So the requires limit is $\lim_{t\to 0}\frac{\sin(t)}{t} = 1$

Q.24 $\lim_{n\to\infty}\left(\sqrt{n^2+n} - \sqrt{n^2+1}\right)$ is________.

[IN, 2016 : 1 Mark]

Solution:

$$\underset{n\to\infty}{Lt}\ \sqrt{n^2+n} - \sqrt{n^2+1}$$

$$= \underset{n\to\infty}{Lt}\ \frac{\sqrt{n^2+n} - \sqrt{n^2+1}}{\left(\sqrt{n^2+n}+\sqrt{n^2+1}\right)}\left(\sqrt{n^2+n}+\sqrt{n^2+1}\right)$$

$$= \underset{n\to\infty}{Lt}\ \frac{n^2+n-n^2-1}{\sqrt{n^2+n}+\sqrt{n^2+1}}$$

$$= \underset{n\to\infty}{Lt}\ \frac{n-1}{n\sqrt{1+\frac{1}{n}}+n\sqrt{1+\frac{1}{n^2}}}$$

$$= \underset{n\to\infty}{Lt}\ \frac{n\left(1-\frac{1}{n}\right)}{n\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{1}{n^2}}\right)} = \frac{1}{1+1} = \frac{1}{2}$$

Q.25 $\underset{x\to 0}{Lt}\dfrac{\log_e(1+4x)}{e^{3x}-1}$ is equal to

(a) 0
(b) $\dfrac{1}{12}$
(c) $\dfrac{4}{3}$
(d) 1

[ME, 2016 : 1 Mark, Set-3]

Solution: (c)

$$\underset{x\to 0}{Lt}\frac{\ln(1+4x)}{e^{3x}-1}\quad \text{0/0 form}$$

$$\underset{x\to 0}{Lt}\frac{\frac{1}{1+4x}\cdot 4}{3e^{3x}}=\frac{4}{3}$$

Q.26 $\lim\limits_{x\to\infty}\sqrt{x_2+x-1}-x$ is

(a) 0
(b) ∞
(c) 1/2
(d) $-\infty$

[ME, 2016 : 2 Marks, Set-3]

Solution: (c)

$$\underset{x\to\infty}{Lt}\sqrt{x^2+x-1}-x$$

$$\underset{x\to\infty}{Lt}\frac{\left(\sqrt{x^2+x-1}-x\right)\left(\sqrt{x^2+x-1}+x\right)}{\sqrt{x^2+x-1}+x}$$

$$\underset{x\to\infty}{Lt}\frac{x^2+x-1-x^2}{\sqrt{x^2+x+1}+x}$$

$$\underset{x\to\infty}{Lt}\frac{x-1}{x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+x}$$

$$\underset{x\to\infty}{Lt}\frac{1-\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}=\frac{1}{1+1}=\frac{1}{2}$$

Q.27 What is the value of $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{xy}{x^2+y^2}$?

(a) 1
(b) −1
(c) 0
(d) Limit does not exit

[CE, 2016 : 1 Mark, Set-II]

Solution: (d)

(i) $$\lim_{x \to \infty} \frac{xy}{x^2+y^2} \lim_{y \to \infty} \left(\frac{0}{0^2+y^2}\right) = 0$$

(i.e., put $x = 0$ and then $y = 0$)

(ii) $$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2} \lim_{x \to 0} \left(\frac{0}{x^2+0}\right) = 0$$

(i.e., put $y = 0$ and then $x = 0$)

(iii) $$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2} \lim_{x \to 0} \frac{x(mx)}{x^2+m^2x^2}$$

(i.e., put $y = mx$)

$$\lim_{x \to \infty} \left(\frac{m}{1+m^2}\right) = \frac{m}{1+m^2}$$

which depends on m.

2.2 CONTINUITY

2.2.1 Definition

A function f(x) is defined for x = a is said to be continuous at x = a if:

1. f(a) i.e., the value of f(x) at x = a is a definite number and
2. the limit of the function f(x) as $x \to a$ exists and is equal to the value of f(x) at x = a.

Note: On comparing the definitions of limit and continuity we find that a function f(x) is continuous at x = a if

$$\underset{x \to a}{\text{Limit}}\ f(x) = f(a)$$

Thus f(x) is continuous at x = a if we have f(a + 0) = f(a − 0) = f(a), otherwise it is discontinuous at x = a.

2.2.2 Continuity from Left and Continuity from Right

Let f be a function defined on an open interval I and let a be any point in I. We say that f is continuous from the left at a, if $\underset{x \to a-0}{\text{Limit}}\ f(x)$ exists and is equal to f(a). Similarly f is said to be continuous from the right at a, if $\underset{x \to a+0}{\text{Limit}}\ f(x)$ exists and is equal to f(a).

∴ A function f(x) is continuous at x = a, Iff it is continuous from left as well as continuous from right.

2.2.3 Continuity in an Open Interval

A function f is said to be continuous in open interval (a, b), iff it is continuous at each point of open interval.

2.2.4 Continuity in a Closed Interval

Let f be a function defined on the closed interval (a, b) f is said to be continuous on the closed interval [a, b] iff it is:

1. continuous from the right at a and
2. continuous from the left at b and
3. continuous on the open interval (a, b).

2.3 DIFFERENTIABILITY

Derivative at a point: Let-I denote the open interval (a, b) in R and let $x_0 \in I$. Then a function $f : I \to R$ is said to be differentiable at x_0, iff:

$$\underset{h \to 0}{\text{Limit}}\left[\frac{f(x_0 + h) - f(x_0)}{h}\right] \text{ or } \underset{x \to x_0}{\text{Limit}}\left[\frac{f(x) - f(x_0)}{x - x_0}\right]$$

exist (finitely) and is denoted by $f'(x_0)$.

2.3.1 Progressive and Regressive Derivatives

The progressive derivative of f (or right derivative of f) at $x = x_0$ is given by

$\underset{h \to 0}{\text{Limit}} \frac{f(x_0 + h) - f(x_0)}{h}$, $h > 0$ and is denoted by $Rf'(x_0)$ or by $f'(x_0 + 0)$ or by $f'(x_0^+)$.

The regressive derivative of f (or left derivative of f) at $x = x_0$ is given by

$\underset{h \to 0}{\text{Limit}} \frac{f(x_0 - h) - f(x_0)}{-h}$, $h > 0$ and is denoted by $Lf'(x_0)$ or by $f'(x_0 - 0)$ or by $f'(x_0^-)$.

2.3.2 Differentiability in an Open Interval

A function f is said to be differentiable in an open interval (a, b), iff it is differentiable at each point of the open interval.

2.3.3 Differentiability in a Closed Interval

A function $f : [a, b] \to R$ is said to be differentiable in closed interval [a, b] iff it is

1. differentiable from right a t a [i.e. R f'(a) exists] and
2. differentiable from left at b [i.e. L f'(a) exists] and
3. differentiable in the open interval (a, b).

2.3.4 Relationship between Differentiability and Continuity

Theorem: If a function is differentiable at any point, then it is necessarily continuous at that point, proof of this theorem follows from definitions of differentiability and continuity.

Note: The converse of this theorem not true.
i.e. Continuity is a necessary but not a sufficient condition for the existence of a finite derivative (differentiability).
i.e. differentiability $\Rightarrow$ continuity
But continuity $\not\Rightarrow$ differentiability

ILLUSTRATIVE EXAMPLES FROM GATE

Q.28 What should be the value of λ such that the function defined below is continuous at $x = \pi/2$?

$$f(x) = \begin{cases} \dfrac{\lambda \cos x}{\frac{\pi}{2} - x} & \text{if } x \neq \pi/2 \\ 1 & \text{if } x = \pi/2 \end{cases}$$

(a) 0
(b) $2/\pi$
(c) 1
(d) $\pi/2$

[CE, GATE-2011, 2 marks]

Solution: (c)

If f(x) is continuous at $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}^+} \frac{\lambda \cos x}{\frac{\pi}{2} - x} = f\left(\frac{\pi}{2}\right) = 1 \quad \text{...(i)}$$

Since the limit is in form of $\frac{0}{0}$, we can use L'hospitals rule on LHS of equation (i) and get

$$\lim_{x \to \frac{\pi}{2}^+} \frac{-\lambda \sin x}{-1} = 1$$

$$\Rightarrow \quad \lambda \sin \frac{\pi}{2} = 1$$

$$\Rightarrow \quad \lambda = 1$$

Q.29 The integrating factor for differential equation $\frac{dP}{dt} + k_2 P = k_1 L_0 e^{-k_1 t}$ is

(a) $e^{-k_1 t}$
(b) $e^{-k_2 t}$
(c) $e^{k_1 t}$
(d) $e^{k_2 t}$ **[CE, GATE-2014 : 1 Mark, Set-2]**

Solution : (d)

Q.30 The function $y = |2 - 3x|$

(a) is continuous $\forall\, x \in R$ and differentiable $\forall\, x \in R$
(b) is continuous $\forall\, x \in R$ and differentiable $\forall\, x \in R$ except at $x = 3/2$
(c) is continuous $\forall\, x \in R$ and differentiable $\forall\, x \in R$ except at $x = 2/3$
(d) is continuous $\forall\, x \in R$ except $x = 3$ and differentiable $\forall\, x \in R$

[ME, GATE-2010, 1 mark]

Solution: (c)

$$y = |2 - 3x| = 2 - 3x \quad 2 - 3x \geq 0$$
$$= 3x - 2 \quad 2 - 3x < 0$$

Therefore,
$$y = 2 - 3x \quad x \leq \frac{2}{3}$$
$$= 3x - 2 \quad x > \frac{2}{3}$$

Since $2 - 3x$ and $3x - 2$ are polynomials, these are continuous at all points. The only concern is at $x = \frac{2}{3}$

Left limit at $x = \frac{2}{3}$ is $2 - 3 \times \frac{2}{3} = 0$

Right limit at $x = \frac{2}{3}$ is $3 \times \frac{2}{3} - 2 = 0$.

$$f\left(\frac{2}{3}\right) = 2 - 3 \times \frac{2}{3} = 0$$

Since, Left limit = Right limit = $f\left(\frac{2}{3}\right)$.

Function is continuous at $\frac{2}{3}$.

y is therefore continuous $\forall\ x \in R$

Now since 2 – 3x and 3x – 2 are polynomials, they are differentiable.

only concern is at $x = \frac{2}{3}$

Now, at $x = \frac{2}{3}$,

$$LD = \text{Left derivative} = -3$$
$$RD = \text{Right derivative} = +3$$
$$LD \neq RD$$

∴ The function y is not differentiable at $x = \frac{2}{3}$

So, we can say that y is differentiable $\forall\ x \in R$, except at $x = \frac{2}{3}$.

Q.31 Consider the function $f(x) = |x|$ in the interval $-1 < x \leq 1$. At the point x = 0, f(x) is

(a) continuous and differentiable
(b) noncontinuous and differentiable
(c) continuous and non-differentiable
(d) neither continuous nor differentiable

[ME, GATE-2012, 1 mark]

Solution: (c)

$$|x| = x \qquad x \geq 0$$
$$= -x \qquad x < 0$$

at $x = 0$, left limit = 0, Right limit = –0 = 0

$$f(0) = 0$$

Since left limit = Right limit = f(0)

So $|x|$ is continuous at $x = 0$

Now,

$$LD = \text{Left derivative (at } x = 0) = -1$$
$$RD = \text{Right derivative (at } x = 0) = +1$$
$$LD \neq RD$$

So $|x|$ is not differentiable at $x = 0$

So $|x|$ is continuous and non-differentiable at x = 0.

Q.32 If a function is continuous at a point,

(a) the limit of the function may not exist at the point.
(b) the function must be derivable at the point.
(c) the limit of the function at the point tends to infinity.
(d) the limit must exist at the point and the value of limit should be same as the value of the function at that point.

[ME, GATE-2014 : 1 Mark, Set-3]

Solution : (d)

f(x) is continuous at any point

if $$\underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to a^+}{\text{Lt}} f(x) = f(a)$$

Q.33 Which one of the following functions is continuous at x = 3?

(a) $f(x) = \begin{cases} 2, & \text{if } x = 3 \\ x-1, & \text{if } x > 3 \\ \dfrac{x+3}{3}, & \text{if } x < 3 \end{cases}$

(b) $f(x) = \begin{cases} 4, & \text{if } x = 3 \\ 8-x & \text{if } x \neq 3 \end{cases}$

(c) $f(x) = \begin{cases} x+3, & \text{if } x \leq 3 \\ x-4 & \text{if } x > 3 \end{cases}$

(d) $f(x) = \dfrac{1}{x^3 - 27}$, if $x \neq 3$

[CS, GATE-2013, 1 Mark]

Solution: (a)

$$\begin{cases} 2, & \text{if } x = 3 \\ x-1, & \text{if } x > 3 \\ \dfrac{x+3}{3}, & \text{if } x < 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+3}{3} = 2$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x-1 = 2$$

Also, $$f(3) = 2$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

So it is continuous at x = 3

option (a) is correct.

Q.34 A function $f(x)$ is continuous in the interval [0, 2]. It is known that $f(0) = f(2) = -1$ and $f(1) = 1$. Which one of the following statements must be true?

(a) There exists a y in the interval (0, 1) such that $f(y) = f(y + 1)$
(b) For every y in the interval (0, 1), $f(y) = f(2 - y)$
(c) The maximum value of the function in the interval (0, 2) is 1
(d) There exists a y in the interval (0, 1) such that $f(y) = -f(2 - y)$

Solution : (a) [CS, GATE-2014 : 2 Marks, Set-1]

(a) As $y \in (0, 1)$; f(y) varies from −1 to 1 similarly f(y + 1) varies from +1 to −1

∴ Let, $g(x) = f(y) - f(y + 1) : y \in (0, 1)$

we get, $g(x) = 0$ for some value of x

i.e. $f(y) = f(y + 1)$ for some $y \in (0, 1)$

(b) $f(y) = f(2 - y)$ only at y = 0 and y = 1

∴ In (0, 1) we cannot say $f(y) = f(2 - y)$

(c) We cannot conclude that the maximum value of f(y) is 1 in (0, 2)

(d) As $y \in (0, 1)$; f(y) varies from –1 to 1 and –f(2 – y) varies from 1 to –1

$\therefore$ Let $g(x) = f(y) + f(2 - y)$; $y \in (0, 1)$

$\therefore$ $g(x) = 0$ for same value of x

i.e. $f(y) = -f(2 - y)$ for some $y \in (0, 1)$

But the difference between y and 2 – y should be less than the length of the interval 2 is not possible.

Q.35 Let the function

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

where $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $f'(\theta)$ denote the derivative of f with respect to θ. Which of the following statements is/are TRUE?

(I) There exists $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) = 0$.

(II) There exists $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) \neq 0$.

(a) I only
(b) II only
(c) Both I and II
(d) Neither I nor II

[CS, GATE-2014 : 1 Mark, Set-1]

Solution : (c)

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

$$f(\pi/6) = 0$$

Since if we put $\theta = \pi/6$ in above determinant it will evaluate to zero, since I and II row will become same.

$$f(\pi/3) = 0$$

Since if we put $\theta = \pi/3$ in above determinant it will evaluate to zero, since I and III row will become same.

So $f(\pi/6) = f(\pi/3)$. Also in the interval $[\pi/6, \pi/3]$ the function $f(\theta)$ is continuous and differentiable (*note* that the given interval doesn't contain any odd multiple of $\pi/2$ where $\tan\theta$ is neither continuous nor differentiable).

Since all the three conditions of **Roll's theorem** are satisfied the conclusion of Rolls theorem is true i.e.

I: $\exists\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) = 0$ is true

Now the statement

II: $\exists\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) \neq 0$

is also true, since the only way it can be false is if $f'(\theta) = 0$ for all values of θ, which is possible only if $f(\theta)$ is a constant which is untrue.

Therefore, both (I) and (II) are correct.

Q.36 The function $f(x) = x \sin x$ satisfies the following equation: $f''(x) + f(x) + t\cos x = 0$. The value of t is ________.

[CS, GATE-2014 : 2 Marks]

Solution :

$$f(x) = x \sin x$$
$$f'(x) = x\cos x + \sin x$$
$$f''(x) = (-x\sin x + \cos x) + \cos x$$
$$f''(x) + f(x) + t\cos x = 0$$
$$\Rightarrow \quad -x\sin x + \cos x + \cos x + x\sin x + t\cos x = 0$$
$$\Rightarrow \quad (2 + t)\cos x = 0$$
$$\Rightarrow \quad t + 2 = 0$$
$$\Rightarrow \quad t = -2$$

Q.37 Let $f(x) = x^{-(1/3)}$ and A denote the area of the region bounded by $f(x)$ and the X-axis, when x varies from –1 to 1. Which of the following statements is/are True?

1. f is continuous in [–1, 1] 2. f is not bounded in [–1, 1] 3. A is nonzero and finite

(a) 2 only (b) 3 only
(c) 2 and 3 only (d) 1, 2 and 3

[CS, GATE-2015 : 2 Marks, Set-2]

Solution: (c)

$$f(x) = \frac{1}{\sqrt[3]{x}}$$

Statement 1: f is continuous in [–1, 1]. Let us check this statement.
We need to check continuity at x = 0

$$\text{Left limit} = \lim_{x\to 0^-}\frac{1}{\sqrt[3]{x}} = \lim_{h\to 0}\frac{1}{\sqrt[3]{0-h}} = \lim_{h\to 0}\frac{-1}{\sqrt[3]{h}} = -\infty$$

$$\text{Right limit} = \lim_{x\to 0^+}\frac{1}{\sqrt[3]{x}} = \lim_{h\to 0}\frac{1}{\sqrt[3]{0+h}} = \lim_{h\to 0}\frac{1}{\sqrt[3]{h}} = +\infty$$

Left limit ≠ Right limit
∴ Statement 1 is false.

Statement 2: f is not bounded is [–1, 1]. Since at x = 0 it goes to $-\infty$ and $+\infty$ the function is not bounded.
∴ Statement 2 is true.

Statement 3: A is non zero and finite.

$$A = \left|\int_{-1}^{0} x^{-1/3}dx\right| + \left|\int_{0}^{1} x^{-1/3}dx\right|$$

$$= \left|\frac{3}{2}\left[x^{2/3}\right]_{-1}^{0}\right| + \left|\frac{3}{2}\left[x^{2/3}\right]_{0}^{1}\right| = \left|\frac{3}{2}\right| + \left|\frac{3}{2}\right| = 3$$

So A is non zero and finite.
∴ Statement 3 is true.

Q.38 Given the following statements about a function $f: R \to R$, select the right option:

P: If $f(x)$ is continuous at $x = x_0$, then it is differential at $x = x_0$.
Q: If $f(x)$ is continuous at $x = x_0$, then it may not be differentiable at $x = x_0$.
R: If $f(x)$ is differentiable at $x = x_0$, then it is also continuous at $x = x_0$.

(a) P is true, Q is false, R is false
(b) P is false, Q is true, R is true
(c) P is false, Q is true, R is false
(d) P is true, Q is false, R is true

[EC, 2016 : 1 Mark, Set-1]

Solution: (b)

P : If $f(x)$ is continuous at $x = x_0$, then it is also differentiable at $x = x_0$

Q : If $f(x)$ is continuous at $x = x_0$, then it may or may not be derivable at $x = x_0$

R : If $f(x)$ is differentiable at $x = x_0$, then it is also continuous at $x = x_0$

P is false

Q is true

R is true Option (b) is correct

Q.39 The values of x for which the function

$$f(x) = \frac{x^2 - 3x - 4}{x^2 + 3x - 4}$$

is NOT continuous are

(a) 4 and –1
(b) 4 and 1
(c) – 4 and 1
(d) – 4 and –1

[ME, 2016 : 1 Mark, Set-2]

Solution: (c)

$$f(x) = \frac{x^2 - 3x - 4}{x^2 + 3x - 4} \text{ is not continous}$$

when

$$x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$
$$x = -4, 1$$

2.4 MEAN VALUE THEOREMS

2.4.1 Rolle's Theorem

If a function f(x) is such that:

1. f(x) is continuous in the closed interval $a \le x \le b$ **and**
2. f′(x) exists for every point in the open interval $a < x < b$ **and**
3. f(a) = f(b),

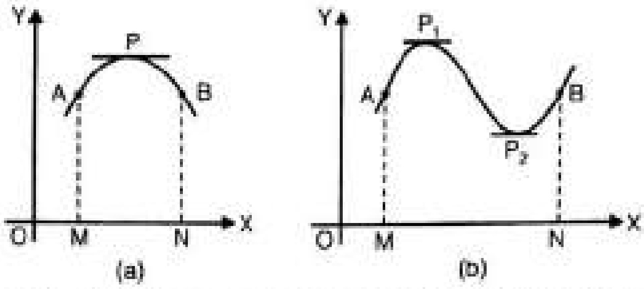
then there exists at least one value of x, say c where $a < c < b$ such that $f'(c) = 0$.

Note: Rolle's theorem will not hold good.

1. If f(x) is discontinuous at some point in the interval $a < x < b$
2. If f′(x) does not exist at some point in the interval $a < x < b$ or
3. If $f(a) \ne f(b)$

2.4.2 Geometrical Interpretation

Let A, B be the points on the curve y = f(x) corresponding to the real numbers a, b, respectively. Since f(x) is continuous in [a, b], the curve y = f(x) has a tangent at each point between A and B. Also as f(a) = f(b) the ordinates of the points A and B are equal i.e. MA = NB [See Figure (a)].



Then rolle's theorem asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to x-axis i.e. there exists atleast one real number c in (a, b) such that $f'(c) = 0$. [see figure (a) above]

There may exist more than one point between A and B, the tangents at which are parallel to x-axis [as shown in Figure (b)] i.e. there exists more than one real number c in (a, b) such that $f'(c) = 0$.

Rolle's theorem ensures the existence of atleast one real number c in (a, b) such that $f'(c) = 0$.

Remarks:

1. Rolle's theorem fails even if one of the three conditions is not satisfied by the function.
2. The converse of Rolle's theorem is not true, since, $f'(x)$ may be zero at a point in (a, b) without satisfying all the three conditions of Rolle's theorem.

ILLUSTRATIVE EXAMPLES

Example: 1

Verify Rolles theorem for the following functions:

(a) $f(x) = x^2 + x - 6$ in $[-3, 2]$

(b) $f(x) = (x - 1)(x - 2)^2$ in $[1, 2]$

(c) $f(x) = (^2 - 1)(x - 2)$ in $[-1, 2]$

Solution:

(a) Given $f(x) = x^2 + x - 6$ ... (i)

(i) As f(x) is a polynomial function, it is continuous in $[-3, 2]$.

(ii) f(x) being a polynomial function is derivable in $(-3, 2)$

(iii) $f(-3) = (-3)^2 - 3 - 6 = 0$, $f(2) = 2^2 + 2 - 6 = 0 \Rightarrow f(-3) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-3, 2)$ such that $f'(x) = 2x + 1$.

Differentiating (i) w.r.t. x, we get $f'(x) = 2x + 1$.

Now $f'(c) = 0 \Rightarrow 2c + 1 = 0 \Rightarrow c = -\frac{1}{2}$.

So there exists $-\frac{1}{2} \in (-3, 2)$ such that $f'\left(-\frac{1}{2}\right) = 0$

Hence, Rolle's theorem is verified.

(b) Given $f(x) = (x - 1)(x - 2)^2$ ... (i)

(i) Since f(x) is a polynomial function, it is continuous is $[1, 2]$.

(ii) f(x) being a polynomial function is derivable in $(1, 2)$.

(iii) $f(1) = (1 - 1)(1 - 2)^2 = 0$, $f(2) = (2 - 1)(2 - 2)^2 = 0 \Rightarrow f(12) = f(2)$

Thus, all the three conditions of Roll's theorem are satisfied, therefore, there exists atleast one real number c in $(1, 2)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = (x-1).2(x-2).1+(x-2)^2.1$$
$$= (x-2)(2x-2+x-2)$$
$$= (x-2)(3x-4)$$

Now $f'(c) = 0$

$\Rightarrow (c-2)(3c-4) = 0$

$\Rightarrow c = 2, 4/3$

But $c \in (1, 2)$, therefore, $c = 4/3$.

So, there exists $(4/3) \in (1, 2)$ such that $f'(4/3) = 0$

Hence, Rolle's theorem is verified.

(c) Given $f(x) = (x^2-1)(x-2)$ ... (i)

(i) Since f(x) is a polynomial function, it is continuous in [−1, 2].

(ii) f(x) being a polynomial function is derivable in (−1, 2).

(iii) $f(-1) = (1-1)(1-2) = 0,\ f(2) = (4-1)(2-2) = 0 \Rightarrow f(-1) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in (−1, 2) such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = (x^2-1).1+(x-2).2x = 3x^2-4x-1.$$

Now $f'(c) = 0 \Rightarrow 3c^2-4c-1 = 0$

$$\Rightarrow c = \frac{4\pm\sqrt{16-4.3(-1)}}{2.3} = \frac{2\pm\sqrt{7}}{3}$$

Also $-1 < \frac{2-\sqrt{7}}{3} < \frac{2+\sqrt{7}}{3} < 2 \Rightarrow \frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ both lie in (−1, 2).

So there exist two real numbers $\frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ in (−1, 2) such that

$$f'\left(\frac{2-\sqrt{7}}{3}\right) = 0 \text{ and } f'\left(\frac{2+\sqrt{7}}{3}\right) = 0$$

Hence, Rolle's theorem is verified.

Example: 2

Verify Rolle's theorem for the following functions and find point (or points) where the derivative vanishes:

$$f(x) = \sin x + \cos x \text{ in } \left[0, \frac{\pi}{2}\right]$$

Solution:

Given: $f(x) = \sin x + \cos x$ ... (i)

(a) f(x) is continuous in $\left[0, \frac{\pi}{2}\right]$

(b) f(x) is derivable in $\left[0, \frac{\pi}{2}\right]$ and

(c) $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$,

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$

Now $f'(c) = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow c = 1$

$\Rightarrow$ $c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, -\frac{3\pi}{4}, \ldots$ but $c \in \left(0, \frac{\pi}{2}\right) \Rightarrow c = \frac{\pi}{4}$.

So there exists $\frac{\pi}{4}$ in $\left(0, \frac{\pi}{2}\right)$ such that $f'\left(\frac{\pi}{4}\right) = 0$.

Hence, Rolle's theorem is verified and $c = \frac{\pi}{4}$.

Example: 3

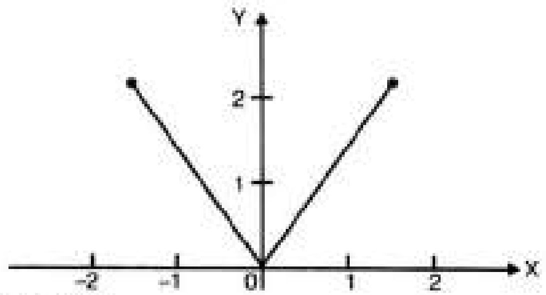
Discuss the applicability of Rolle's theorem for the function $f(x) = |x|$ in $[-2, 2]$.

Solution:

Given: $f(x) = |x|, x \in [-2, 2]$ ...(i)

the graph of $f(x) = |x|$ in $[-2, 2]$

is shown in figure



(a) f(x) is continuous is [–2, 2]

(b) Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{x}{|x|}, x \neq 0$$

$\Rightarrow$ the derivative of f(x) does not exist at x = 0

$\Rightarrow$ f(x) is not derivable in (– 2, 2)

Thus, the condition (ii) of Rolle's theorem is not satisfied, therefore, Rolle's theorem is not applicable to the function $f(x) = |x|$ in $[-2, 2]$.

Moreover, $f(-2) = |-2| = 2$ and $f(2) = |2| = 2 \Rightarrow f(-2) = f(2)$, so the condition (iii) of Rolle's theorem is satisfied.

Further, it is clear from the graph that there is not point of the cure $y = |x|$ i (–2, 2) at which the tangent is parallel to x-axis.

2.4.3 Lagrange's Mean Value Theorem

If a function f(x) is:

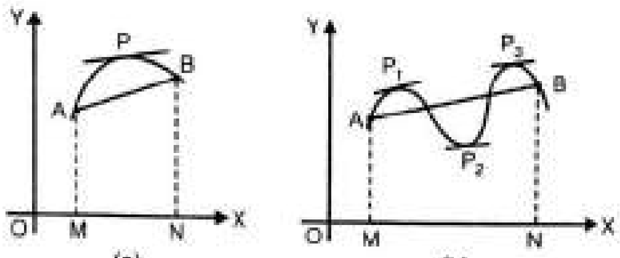
1. Continuous in closed interval $a \le x \le b$ and
2. Differentiable in open interval (a, b) i.e., $a < x < b$,

   then there exist at least one value c of x lying in the open interval $a < x < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2.4.4 Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers a, b respectively. Since f(x) is continuous in [a, b], the graph of the curve $y = f(x)$ is continuous from A to B. Again, as f(x) is derivable in (a, b) the curve $y = f(x)$ has a tangent at each point between A and B. Also as $a \neq b$, the slope of the chord AB exists and the slope of the chord AB = $\frac{f(b)-f(a)}{b-a}$.



(a) (b)

Then Lagrange's Mean Value Theorem asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to the chord AB. There may exist more than one point between A and B the tangents at which are parallel to the chord AB [as shown in Figure (b)]. Lagrange's mean value theorem ensures the existence of atleast one real number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Remarks:

1. Lagrange's mean value theorem fails for the function which does not satisfy even one of the two conditions.
2. The converse of Lagrange's mean value theorem may not be true, for, f'(c) may be equal to $\frac{f(b)-f(a)}{b-a}$ at a point c in (a, b) without satisfying both the conditions of Lagrange's mean value theorem.

ILLUSTRATIVE EXAMPLES

Example: 1

Verify Lagrange's mean mean value theorem for the following functions in the given interval and find 'c' of this theorem.

(a) $f(x) = x^2 + 2x + 3$ in [4, 6]

(b) $f(x) = px^2 + qx + r$, $p \neq 0$, in [a, b]

Solution:

(a) Given $f(x) = x^2 + 2x + 3$

(i) f(x) being a polynomial function is continuous in [4, 6]. ... (i)

(ii) f(x) being a polynomial function is derivable in (4, 6).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in (4, 6) such that

$$f'(c) = \frac{f(6)-f(4)}{6-4}$$

$$f(6) = 6^2 + 2.6 + 3 = 51,\ f(4) = 4^2 + 2.4 + 3 = 27.$$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2x + 2 \Rightarrow f'(c) = 2c + 2$$

$\therefore \quad f'(c) = \frac{f(6)-f(4)}{6-4} \quad 2c+2 = \frac{51-27}{2} \Rightarrow 2c+2 = 12$

$\Rightarrow \quad 2c = 10 \Rightarrow c = 5$

Thus, there exists $c = 5$ in $(4, 6)$ such that $f'(5) = \frac{f(6)-f(4)}{6-4}$

Hence, Lagrange's mean value theorem is verified and $c = 5$.

(b) Given $f(x) = px^2 + qx + r,\ p \neq 0$

(i) f being a polynomial function is continuous in [a, b]

(ii) f being a polynomial function is derivable in (a, b).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in (a, b) such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

$$f(b) = pb^2 + qb + r,\ f(a) = pa^2 + qa + r.$$

Diffeentiating (1) w.r.t. x, we get

$$f'(x) = 2px + q \Rightarrow f'(c) = 2pc + q.$$

$$\therefore \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 2pc + q = \frac{(pb^2+qb+r)-(pa^2+qa+r)}{b-a}$$

$$\Rightarrow \quad 2pc + q = \frac{p(b^2+a^2)+q(b-a)}{b-a}$$

$$\Rightarrow \quad 2pc = p(a+b)$$

$$\Rightarrow \quad c = \frac{a+b}{2} \text{ and } \frac{a+b}{2} \in (a, b)$$

Thus, there exist $c = \frac{a+b}{2}$ in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Hence Lagrange's mean value theorem is verified and $c = \frac{a+b}{2}$

Example: 2

Find a point on the graph of $y = x^3$ where the tangent is parallel to the chord joining (1, 1)and (3, 27).

Solution:

$$f(x) = x^3 \text{ in the interval } [1, 3]$$

(a) f(x) being a polynomial is continuous in [1, 3].

(b) f(x) being a polynomial is derivable in (1, 3).

Thus, both the conditions of Lagrange's mean value theorem are satisfied by the function f(x) in [1, 3], therefore, there exists atleast one real number c in (1, 3) such that

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

$$f(3) = 3^3 = 27 \text{ and } f(1) = 1^3 = 1.$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2.$$

Now $f'(c) = \frac{f(3)-f(1)}{3-1} \Rightarrow 3c^2 = \frac{27-1}{3-1} \Rightarrow 3c^2 = 13$

$\Rightarrow \quad c^2 = \frac{13}{3} = \frac{39}{9}$

$\Rightarrow \quad c = \pm\frac{\sqrt{39}}{3}$

But $\quad c \in (1, 3) \Rightarrow c = \frac{\sqrt{39}}{3}$

When $\quad x = \frac{\sqrt{39}}{3}$, from (1) $y = \frac{\sqrt{39}}{3}$

Hence, there exists a point $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ on the given curve $y = x^3$ where the tangent is parallel to the chord joining the points (1, 1) and (3, 27).

Example: 3

Does the Lagrange's mean value theorem apply to $f(x) = x^{1/3}, -1 \le x \le 1$? What conclusions can be drawn?

Solution:

Given, $\quad f(x) = x^{1/3}, x \in [-1, 1]$ ... (i)

(a) f(x) is continuous in [– 1, 1]

(b) Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}, x \neq 0 \quad \text{... (i)}$$

$\Rightarrow$ The derivative of f(x) does not exist at x = 0

$\Rightarrow$ f(x) is not derivable in (–1, 1).

Thus, the condition (ii) of lagrange's mean value theorem is not satisfied by the function $f(x) = x^{1/3}$ in [–1, 1] and hence Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in [– 1, 1] and hence Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in [–1, 1].

Conclusion. However, from (2), $f'(c) = \frac{1}{3c^{2/3}}, c \neq 0$

Also $f(-1) = (-1)^{1/3} = -1$, $f(1) = 1^{1/3} = 1$ (we have taken only real values)

$\therefore \quad f'(c) = \frac{f(1)-f(-1)}{1-(-1)}$

$\Rightarrow \quad \frac{1}{3c^{2/3}} = \frac{1-(-1)}{1-(-1)} = \frac{2}{2} = 1$

$\Rightarrow \quad c^{2/3} = \frac{1}{3} \Rightarrow c^2 = \frac{1}{27} \Rightarrow c = \pm\frac{1}{3\sqrt{3}}$.

As $-1 < -\frac{1}{3\sqrt{3}} < \frac{1}{3\sqrt{3}} < 1 \Rightarrow c = \pm\frac{1}{\sqrt{3}}$ both lie in (–1, 1)

Thus, we find that there exist two real numbers $c = \pm\frac{1}{\sqrt{3}}$ in (–1, 1) such that $f'(c) = \frac{f(1)-f(-1)}{1-(-1)}$.

If follows that the converse of Lagrange's mean value theorem may not be true.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.40 A rail engine accelerates from its stationary position for 8 seconds and travels a distance of 280 m. According to the Mean Value Theorem, the speedometer at a certain time during acceleration must read exactly

(a) 0 (b) 8 kmph
(c) 75 kmph (d) 126 kmph

[CE, GATE-2005, 2 marks]

Solution: (d)

Since the position of rail engine S(t) is continuous and differentiable function, according to Lagranges mean value theorem

$\exists t$ where $0 \le t \le 8$ such that

$$S'(t) = v(t) = \frac{S(8) - S(0)}{8 - 0} = \frac{(280 - 0)}{(8 - 0)} \text{ m/sec}$$

$$= \frac{280}{8} \text{ m/sec} = \frac{280}{8} \times \frac{3600}{1000} \text{ kmph} = 126 \text{ kmph}$$

where v(t) is the velocity of the rail engine.

Q.41 A function $f(x) = 1 - x^2 + x^3$ is defined in the closed interval [−1, 1]. The value of x in the open interval (−1, 1) for which the mean value theorem is satisfied, is

(a) −1/2 (b) −1/3
(c) 1/3 (d) 1/2

[EC, GATE-2015 : 1 Mark, Set-1]

Solution: (b)

Since $f(1) \ne f(-1)$, Roll's mean value theorem does not apply.

By Lagrange mean value theorem

$$f'(x) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2}{2} = 1$$

$$-2x + 3x^2 = 1$$

$$x = 1, -\frac{1}{3}$$

x lies in (−1, 1) $\Rightarrow$ $x = -\frac{1}{3}$

2.4.5 Some applications of Lagrange's Mean Value theorem

1. If a function f(x) is
   (a) continuous in [a, b]
   (b) derivable in (a, b) and
   (c) $f'(x) > 0$ for all x in (a, b), then f(x) is strictly increasing function in [a, b].

Proof. Let x_1, x_2 be any two members of [a, b] such that $a \le x_1 < x_2 \le b$ then f(x) satisfied both the conditions of Lagrange's mean value theorem in $[x_1, x_2]$, therefore, there exists atleast one real number c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow \quad (x_2 - x_1) f'(c) = f(x_2) - f(x_1)$$

But $f'(x) > 0$ for all x in (a, b) $\Rightarrow f'(c) > 0$ for all c in (x_1, x_2). Also $x_1 < x_2$ ile. $x_2 - x_1 > 0$

$$\Rightarrow \quad (x_2 - x_1) f'(c) > 0$$

$$\Rightarrow \quad f(x_2) - f(x_1) > 0$$

$\Rightarrow f(x_2) > f(x_1)$, for all x_1, x_2 such that $a \le x_1 < x_2 \le b$.
Hence, f(x) is strictly increasing in [a, b]

2. If a function f(x) is
   (a) continuous in [a, b]
   (b) derivable in (a, b)
   (c) $f'(x) < 0$ for all x in (a, b), then f(x) is strictly decreasing function in [a, b].
   (For the proof, proceed as above)

2.4.6 Some Important Deductions from Mean Value Theorems

1. If a function f(x) be such that f'(x) is zero throughout the interval, then f(x) must be constant throughout the interval.
2. If f(x) and $\phi(x)$ be two functions such that $f'(x) = \phi'(x)$ throughout the interval (a, b), then f(x) and $\phi(x)$ differ only by a constant.
3. If f'(x) is:
   (a) continuous in closed interval [a, b]
   (b) differentiable in open interval (a, b)
   (c) f'(x) is –ve in $a < x < b$, then f(x) is monotonically decreasing function in the closed interval [a, b] and f'(x) is positive in $a < x < b$, then f(x) is monotonically increasing function in the closed interval [a, b].

2.4.7 Some Standard Results on Continuity and Differentiability of Commonly used Functions

It is important to remember the following facts regarding common functions while checking applicability of Rolle's and Lagrange's mean value theorems:

1. Constant function is differentiable everywhere [$f'(x) = 0$, $\forall x$].
2. Any polynomial function is continuous and differentiable everywhere.
3. The exponential function (e^x, a^x etc), sin x, as well as cos x are also continuous and differentiable everywhere.
4. log function, trigonometric and inverse trigonometric functions are differentiable within their domains.
5. tan x is discontinuous at $x = \pm \pi/2, \pm 3\pi/2., \ldots$
6. |x| is continuous but not differentiable at x = 0.
7. If $f'(x) \to \pm \infty$ as $x \to k$, then that function is not differentiable at x = k.
8. Sum, difference, product, quotient and compositions of continuous and differentiable functions are continuous and differentiable.

2.5 COMPUTING THE DERIVATIVE

Rules of Differentiation:

$$(f + g)' = f' + g' \qquad \text{(Sum rule)}$$

$$(f - g)' = f' - g' \qquad \text{(Difference rule)}$$

$$(f.g)' = fg' + gf' \qquad \text{(Product rule)}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \qquad \text{(Quotient rule)}$$

$$\frac{1}{dx}(f(g(x))) = \frac{df}{dg}\cdot\frac{dg}{dx} \qquad \text{(Chain rule)}$$

Using the above five rules, we can differentiate most of the cases where y is an explicit function of x.

The following is the table of derivatives of commonly occuring functions:

f(x)	f'(x)
x^n	$n\,x^{n-1}$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\log_a e.\left(\frac{1}{x}\right)$
e^x	e^x
a^x	$a^x \log_e a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec } x$	$-\text{cosec } x \cot x$
$\cot x$	$-\text{cosec}^2 x$
$\sin h\, x$	$\cos h\, x$

f(x)	f'(x)
$\cos h\, x$	$\sin h\, x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\text{cosec}^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\cot^{-1} x$	$\frac{1}{1+x^2}$
$\lvert x\rvert$	$\frac{x}{\lvert x\rvert}\ (x \neq 0)$

Most explicit functions can be differentiated by using above table along with the five rules of differentiation. For more complicates cases, we have to resort to more advanced methods of differentiation as given below:

1. Differentiation by substitution
2. Implicit differentiation
3. Logarithmic differentiation
4. Parametric differentiation

2.5.1 Differentiation by Substitution

There are no hard and fast rules for making suitable substitutions. It is the experience which guides us for the selection of a proper substitution. However, some useful suggestions are given below:

If the function contains an expression of the form

1. $a^2 - x^2$, put $x = a \sin t$ or $x = a \cos t$
2. $a^2 + x^2$, put $x = a \tan t$ or $x = a \cot t$
3. $x^2 - a^2$, put $x = a \sec t$ or $x = a \text{ cosec } t$
4. $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$, put $x = a \cos t$
5. $a \cos x \pm b \sin x$, put $a = r\cos\theta$ and $b = r\sin\theta$, $r > 0$.

ILLUSTRATIVE EXAMPLES

Example:

Differentiate the following functions (by suitable substitutions) w.r.t. x.

(a) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(b) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$

(c) $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

(d) $\tan^{-1}\left(\sqrt{1+x^2}+x\right)$

Solution:

(a) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, put $x = \tan\theta$ i.e. $\theta = \tan^{-1}x$.

then $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin(2\theta)) = 2\theta$

$= 2\tan^{-1}x$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

(b) Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$, put $x = \tan\theta$ i.e. $\theta = \tan^{-1}x$,

then $y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}+1}{\tan\theta}\right)$

$$= \tan^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right) = \tan^{-1}\left[\frac{\frac{1}{\cos\theta}+1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)$$

$= \frac{\pi}{2}-\frac{\theta}{2} = \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0-\frac{1}{2}\cdot\frac{1}{1+x^2} = -\frac{1}{2(1+x^2)}$$

(c) Let $y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{x-\frac{1}{x}}{x+\frac{1}{x}}\right) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

put $x = \tan\theta$ i.e. $\theta = \tan^{-1}x$,

then $y = \cos^{-1}\left(\frac{\tan^2\theta-1}{\tan^2\theta+1}\right) = \cos^{-1}\left(\frac{-1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}(\cos(\pi-2\theta))$

$= \pi-2\theta = \pi-2\tan^{-1}x$, differentiating w.r.t., x, we get

$$\frac{dy}{dx} = 0-2.\frac{1}{1+x^2} = -\frac{2}{1+x^2}$$

(d) Let $y = \tan^{-1}\left(\sqrt{1+x^2}+x\right)$

put $x = \cot\theta$ i.e. $\theta = \cot^{-1} x$

then
$$y = \tan^{-1}\left(\sqrt{1+\cot^2\theta} + \cot\theta\right)$$
$$= \tan^{-1}(\text{cosec}\,\theta + \cot\theta)$$
$$= \tan^{-1}\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)$$
$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\cot\frac{\theta}{2}\right)$$
$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) = \frac{\pi}{2} - \frac{\theta}{2}$$
$$= \frac{\pi}{2} - \frac{1}{2}\cot^{-1} x$$, differentiating w.r.t. x, we get
$$\frac{dy}{dx} = 0 - \frac{1}{2}\left(-\frac{1}{1+x^2}\right) = \frac{1}{2(1+x^2)}$$

2.5.2 Implicit Differentiation

If y be a function of x defined by an equation such as
$$y = 7x^4 - 5x^3 + 11x^2 + \sqrt{2}x - 3 \quad \text{... (i)}$$
y is said to be defined explicitly in terms of x and we write y = f(x) where
$$f(x) = 7x^4 - 5x^3 + 11x^2 + \sqrt{2}x - 3$$
However, if x and y are connected by an equation of the form
$$x^4y^3 - 3x^3y^5 + 7y^3 - 8x^2 + 9 = 0 \quad \text{... (ii)}$$
i.e. f(x, y) = 0, then y cannot be expressed explicitly in terms of x. But, still the value of y depends upon that of x and there may exist one or more functions 'f' connecting y with x so as to satisfy equation (ii) or there may not exist any of the functions satisfying equation (ii).

For example, consider the equations
$$x^2 + y^2 - 25 = 0 \quad \text{... (iii)}$$
and
$$x^2 + y^2 + 25 = 0 \quad \text{... (iv)}$$
In equation (ii), y may be expressed explicitly in terms of x, but y is not a function of x. Here we have two functions of x (or two functions of y if y were considered to be independent variable) f_1 and f_2 defined by $f_1(x) = \sqrt{25-x^2}$ and $f_2(x) = -\sqrt{25-x^2}$ which satisfy equation (iii).

In equation (iv), there are no real values of x that can satisfy it.

In cases (ii), (iii) and (iv), we say that y is an implicit function of x (or x is an implicit function of y) and in all such cases, we find the derivative of y with regard to x (or the derivative of x with regard to y) by the process called implication differentiation. Of course, wherever we differentiate implicitly an equation that defines one variable as an implicit function of another variable, we shall assume that the function is differentiable.

ILLUSTRATIVE EXAMPLES

Example: 1

Find $\frac{dy}{dx}$ when $x^2 + xy + y^2 = 100$.

Solution:

Given, $x^2 + xy + y^2 = 100$

Keeping in mind that y is a function of x, differentiating both sides w.r.t. x, we get

$$2x + \left(x.\frac{dy}{dx} + y.1\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x + 2y)\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Example: 2

If $x^{2/3} + y^{2/3} = a^{2/3}$, find $\frac{dy}{dx}$.

Solution:

Given, $x^{2/3} + y^{2/3} = a^{2/3}$ ... (i)

Differentiating both sides of (i) w.r.t. x, regarding y as a function of x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{x^{1/3}} + \frac{1}{y^{1/3}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}$$

Example: 3

If $\sin^2 y + \cos xy = \pi$, find $\frac{dy}{dx}$.

Solution:

Given, $\sin^2 y + \cos xy = \pi$

Differentiating both sides of (i) w.r.t. x, regarding y as as function of x, we get

$$2(\sin y)\cdot\cos y\cdot\frac{dy}{dx} - \sin xy\cdot\left(x\frac{dy}{dx} + y.1\right) = 0$$

$$\Rightarrow \quad (2\sin y\cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \quad (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Example: 4

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + ...\text{to } \infty}}}$, prove that $(1-2y)\frac{dy}{dx} = \sin x$.

Solution:

Given,

$$y = \sqrt{\cos x + y}$$

$\Rightarrow$
$$y^2 = \cos x + y$$

$\Rightarrow$
$$y^2 - y = \cos x$$

differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} - \frac{dy}{dx} = -\sin x$$

$\Rightarrow$
$$(1-2y)\frac{dy}{dx} = \sin x$$

2.5.3 Logarithmic Differentiation

In order to simplify the differentiation of some functions, we first take logarithms and then differentiate. Such a process is called logarithmic differentiation. This is usually done in two types of problems.

1. When the given function is a product of some functions, then the logarithm converts the product into a sum and this facilitates the differentiation.
2. When the variable occurs in the exponent i.e. the given function is of the form $[f(x)]^{\phi(x)}$.

Derivative of u^v where u, v are differentiable functions of x

Let

$$y = u^v, \text{ taking logarithm of both sides, we get}$$

$$\log y = v \log u, \text{ differentiating w.r.t. x, we get}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(v \log u)$$

$\Rightarrow$
$$\frac{dy}{dx} = y\frac{d}{dx}(v \log u) = u^v \frac{d}{dx}(v \log u)$$

ILLUSTRATIVE EXAMPLES

Example: 1

Differentiate the following functions w.r.t. x:

(a) x^x

(b) $\cos(x^x)$.

Solution:

(a) Let
$$y = x^x,$$

Taking logarithm of both sides, we get

$$\log y = x \log x.$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x . 1$$

$\Rightarrow$
$$\frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$$

(b) Let
$$y = \cos(x^x), \text{ differentiating w.r.t. x, we get}$$

$$\frac{dy}{dx} = -\sin(x^x).\frac{d}{dx}(x^x)$$

Now $\frac{d}{dx}(x^x)$ has been obtained preciously in part (a).

So, $$\frac{dy}{dx} = -\sin(x^x).\ x^x(1 + \log x)$$

Example: 2

If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution:

Given, $x^y = e^{x-y}$, taking logarithm of both sides, we get

$$y \log x = (x - y) \log e = (x - y).\ 1 = x - y$$

$\Rightarrow$ $$y + y \log x = x$$

$\Rightarrow$ $$(1 + \log x)\ y = x$$

$\Rightarrow$ $$y = \frac{x}{1+\log x},$$ differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2} = \frac{1+\log x-1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

2.5.4 Derivatives of Functions in Parametric forms

If x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e. if $x = f(t)$ and $y = g(t)$ then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

OR $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \left(\text{provide } \frac{dx}{dt} \neq 0\right)$$

ILLUSTRATIVE EXAMPLES

Example: 1

If $x = a(t + \sin t)$, $y = a(1 - \cos t)$, fing $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$.

Solution:

Given, $$x = a(t + \sin t) \text{ and } y = a(1 - \cos t)$$

Differentiating both w.r.t. t, we get

$$\frac{dx}{dt} = a(1 + \cos t)$$

and $$\frac{dy}{dt} = a\,(0 - (-\sin t)) = a \sin t.$$

We know that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\therefore$ $\frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$.

$\therefore$ $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} = \tan\frac{\pi}{4} = 1.$

Example: 2

Differentiate $\frac{x^3}{1-x^3}$ w.r.t. x^3.

Solution:

Let $y = \frac{x^3}{1-x^3}$ and $z = x^3$ so that $\frac{dy}{dz}$ is wanted.

Differentiating both w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1-x^3).3x^2 - x^3.(0-3x^2)}{(1-x^3)^2} = \frac{3x^2}{(1-x^3)^2},$$

and $\frac{dz}{dx} = 3x^2.$

We know that $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$

$\therefore$ $\frac{dy}{dz} = \frac{3x^2}{(1-x^3)^2}\times\frac{1}{3x^2} = \frac{1}{(1-x^3)^2}, x \neq 1$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.42 If $x = a(\theta + \sin\theta)$ and $y = a(1-\cos\theta)$, then dy/dx will be equal to

(a) $\sin\left(\frac{\theta}{2}\right)$

(b) $\cos\left(\frac{\theta}{2}\right)$

(c) $\tan\left(\frac{\theta}{2}\right)$

(d) $\cot\left(\frac{\theta}{2}\right)$

[ME, GATE-2004, 1 mark]

Solution: (c)

Given, $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$

$\therefore$ $\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2a\sin\left(\frac{\theta}{2}\right).\cos\left(\frac{\theta}{2}\right)}{a\times 2\cos^2\left(\frac{\theta}{2}\right)} = \tan\theta/2$$

Q.43 If $y = f(x)$ is the solution of $\frac{d^2y}{dx^2} = 0$, with the boundary conditions $y = 5$ at $x = 0$, and $\frac{dy}{dx} = 2$ at $x = 10$, $f(15) =$ ______.

[EC, GATE-2014 : 2 Marks, Set-2]

Solution :

$$\frac{d^2y}{dx^2} = 0$$

$$\frac{dy}{dx} = C_1$$

$$\Rightarrow \quad C_1 = 2$$

$$y = C_1x + C_2$$

at $x = 0$

$$y = 5 = C_2$$

$$\therefore \quad y = 2x + 5$$

at $y(15) = 2 \times 15 + 5 = 35$

2.6 APPLICATIONS OF DERIVATIVES

There are two areas where derivatives are used

1. Increasing and Decreasing Functions
2. Maxima and Minima
   (a) Relative maxima and minima
   (b) Absolute maxima and minima
3. Taylor's and Maclaurin's Series Expansion of Functions
4. Slope determination of line

2.6.1 Increasing and Decreasing Functions

Let f be a real valued function defined in an interval D (a subset of R), then f is called an increasing function in an interval D_1(a subset of D) iff

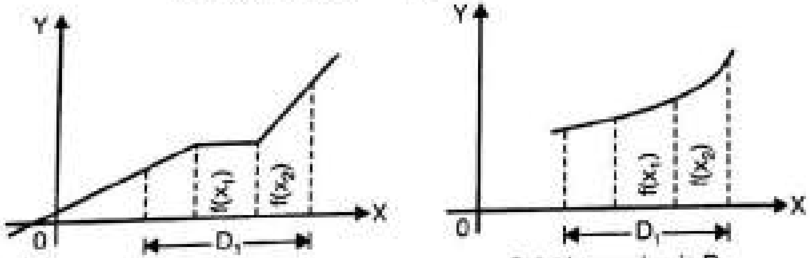
for all $x_1, x_2 \in D_1$,

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$$

and f is called a strict increasing function (or monotonically increasing function) in D_1 iff

for all $x_1, x_2 \in D_1$,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$



Increasing in D_1

(a)

Strict increasing in D_1

(b)

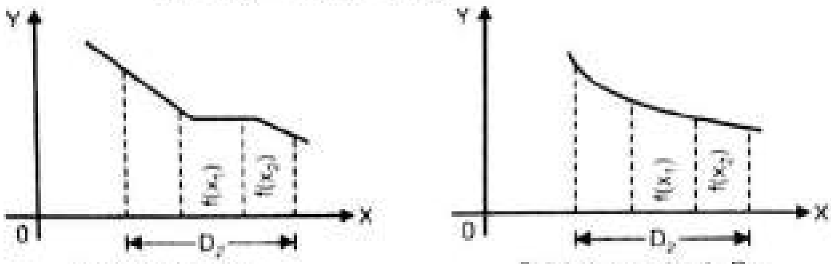
Analogously, f is called a decreasing function in an interval D_2 (a subset of D) iff

for all $x_1, x_2 \in D_2$,

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$$

and f is called a strict decreasing function (or monotonically decreasing function) in D_2 iff for all $x_1, x_2 \in D_2$,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

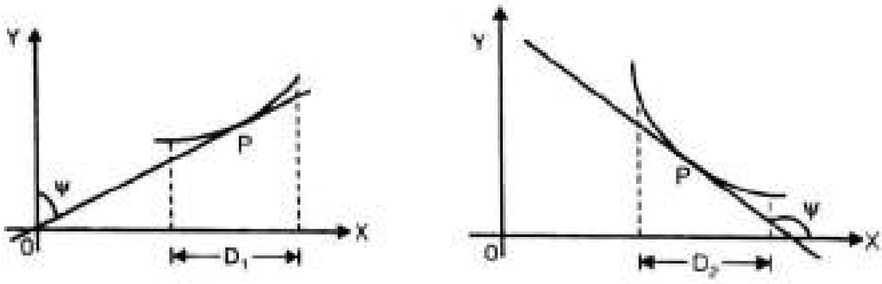


Decreasing in D_2 (a) Strict decreasing in D_2 (b)

2.6.1.1 Conditions for an Increasing or a Decreasing Function

Now we shall see how to use derivative of a function to determine where it is increasing and where it is decreasing.

We know that the derivative (if it exists) at a point P of a curve represents the slope of the tangent to the curve at P.

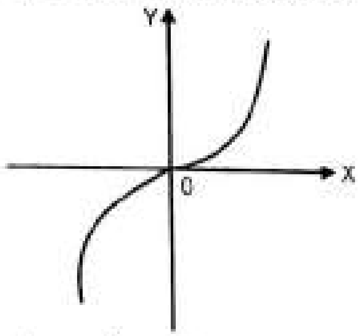


(a) (b)

Intuitively, from above fig. (i) we see that if f is a strict increasing function in D_1 (a subset of D_f), then the tangent to the curve $y = f(x)$ at every point of D_1 makes an acute angle ψ with the positive direction of x-axis, therefore $\tan\psi > 0 \Rightarrow f'(x) > 0$ for all $x \in D_1$.

Analogously, from above figure (ii) we see that if f is a strict decreasing function in D_2 (a subset of D_f), then the tangent to the curve $y = f(x)$ at every point of D_2 makes obtuse angle ψ with the positive direction of x-axis, therefore, $\tan\psi < 0 \Rightarrow f'(x) < 0$ for all $x \in D_2$.

But this intuition may fail, for example, consider the function $f(x) = x^3$, $D_f = R$.



A portion of its graph is shown in above figure. It is a strict increasing function. However, here $f'(x) = 3x^2$ and at $x = 0$, $f'(0) = 0$, so the slope of the tangent at $x = 0$ is not positive, it is zero.

In fact, we have:

1. If a function f is increasing in D_1(a subset of D_f), then $f'(x) \geq 0$ for all $x \in D_1$.
2. If a function f is decreasing in D_2 (a subset of D_f), then $f'(x) \leq 0$ for all $x \in D_2$.

Conversely, common sense tells us that a function is increasing when its rate of change (derivative) is positive and decreasing when its rate of change is negative. We state these results as follows:

Theorem 1: If a function f is continuous is [a, b], and derivable in (a, b) and

1. $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing in [a, b]
2. $f'(x) > 0$ for all $x \in (a, b)$, then f is strict increasing in [a, b].

Theorem 2: If a function f is continuous in [a, b], and derivable in (a, b) and

1. $f'(x) \leq 0$ for all x in (a, b), then f(x) is decreasing in [a, b].
2. $f'(x) < 0$ for all $x \in (a, b)$, then f(x) is strict decreasing in [a, b].

Remark: The formal proofs of these theorems are based on Lagrange's Mean value Theorem.

Corollary. If a function f(x) is continuous in [a, b], derivable in (a, b) and

1. $f'(x) > 0$ for all x in (a, b) except for a finite number of points where $f'(x) = 0$, then f(x) is strict increasing in [a, b].
2. $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x) = 0$, then f(x) is strict decreasing in [a, b].

ILLUSTRATIVE EXAMPLES

Example: 1

Prove that the function $f(x) = ax + b$ is strictly increasing iff $a > 0$.

Solution:

Given: $f(x) = ax + b, D_f = R.$

Note that f is continuous and differentiable for all $x \in R$.

Differentiating the given function w.r.t. x, we get $f'(x) = a$.

Now the given function is strictly increasing iff $f'(x) > 0$ i.e. iff $a > 0$.

Hence, the given function is strictly increasing for all $x \in R$ iff $a > 0$.

Example: 2

Prove that the function e^{2x} is strictly increasing on R.

Solution:

Let $f(x) = e^{2x}, D_f = R.$

Differentiating w.r.t. x, we get

$$f'(x) = e^{2x}.2 > 0 \text{ for all } x \in R.$$

$\Rightarrow$ f(x) is strictly increasing on R.

Example: 3

Prove that $\frac{2}{x}+5$ is a strictly decreasing function

Solution:

Let $f(x) = \frac{2}{x}+5, D_f = R - [0].$

Diff. it w.r.t. x, we get $f'(x) = 2.(-1.\ x^{-2}) + 0 = -\frac{2}{x^2}$

Since $x^2 > 0$ for all $x \in R, x \neq 0$, therefore,

$f'(x)<0$ for all $x \in R, x \neq 0$, i.e., for all $x \in D_f$

$\Rightarrow$ the given function is strictly decreasing.

Example: 4

Prove that the function $f(x) = x^3 - 6x^2 + 15x - 18$ is strictly increasing on R.

Solution:

Given, $f(x) = x^3 - 6x^2 + 15x - 18,\ D_f = R.$

Diff. it w.r.t. we get $f'(x) = 3x^2 - 6.2\,x + 15.1 = 3(x^2 - 4x + 5)$

$= 3[(x-2)^2 + 1] \geq 3 \quad (\because (x-2)^2 \geq 0 \text{ for all } x \in R)$

$\Rightarrow$ $f'(x) > 0$ for all $x \in R$.

$\Rightarrow$ f(x) is strictly increasing function for all $x \in R$.

Example: 5

Find the intervals in which the following functions are strictly increasing or strictly decreasing

(a) $f(x) = 10 - 6x - 2x^2$

(b) $f(x) = x^2 - 12x^2 + 36x + 17$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

Solution:

(a) Given, $f(x) = 10 - 6x - 2x^2,\ D_f = R.$

Differentiating it w.r.t. x, we get

$$f'(x) = 0 - 6.1 - 2.\,2x = -6 - 4x = -4\left(x + \frac{3}{2}\right).$$

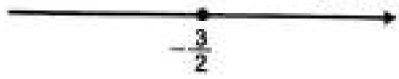
Putting, $f'(x) = 0$, we get $\dfrac{20 \pm \sqrt{400-156}}{2} = 0$

$$\Rightarrow \quad x + \frac{3}{2} = 0$$

$$\Rightarrow \quad x = -\frac{3}{2}$$

So there is only one critical point which is $x = -\frac{3}{2}$

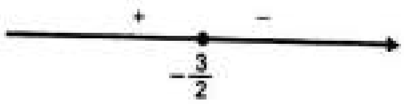
Plotting this critical point on the number line we get the following picture



So the critical point divides the real number line into two regions which are $x \in \left(-\infty, -\frac{3}{2}\right)$ and $x \in \left(-\frac{3}{2}, \infty\right)$

Now we find $f'(0) = -6$ which is negative and so the region $x \in \left(-\frac{3}{2}, \infty\right)$ (which contains $x = 0$) is the region where the function is strictly decreasing.

Therefore in the other region i.e. $x \in \left(-\infty, -\frac{3}{2}\right)$ is the region in which the function is strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.



(b) Given, $f(x) = x^3 - 12x^2 + 36x + 17, D_f = R.$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 24x + 36$$
$$= 3(x^2 - 8x + 12)$$
$$= 3(x-2)(x-6).$$

Putting, $f'(x) = 0$ i.e. $3(x-2)(x-6) = 0$

$\Rightarrow$ $(x-2)(x-6) = 0$

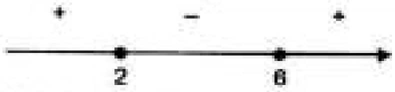
$\Rightarrow$ $x = 2$ or $x = 6$ are the two critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, 2)$ and $x \in (2, 6)$ and $x \in (6, \infty)$.

Now we find $f'(0) = 3(0-2)(0-6) = +36$ which is positive and so in the region $x \in (-\infty, 2)$ (which contains $x = 0$), the function is strictly increasing.

Therefore in the next region i.e. $x \in (2, 6)$, the function is strictly decreasing and in the next region $x \in (6, \infty)$, the function is again strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.

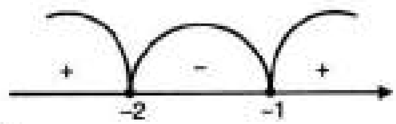


So the final region in which the function strictly increasing is $x \in (-\infty, 2) \cup (6, \infty)$ and the region in which the function is strictly decreasing is $x \in (2, 6)$.

(c) Given, $f(x) = -2x^3 - 9x^2 - 12x + 1, D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = -6x^2 - 18x - 12$$
$$= -6(x^2 + 3x + 2)$$
$$= -6(x+2)(x+1).$$



Putting, $f'(x) = 0$ i.e. $-6(x+2)(x+1) = 0$

$\Rightarrow$ $(x+2)(x+1) = 0$

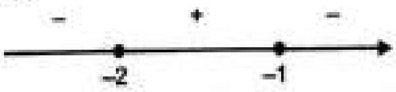
$\Rightarrow$ $x = -2$ and $x = -1$ are the critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, -2)$ and $x \in (-2, -1)$ and $x \in (-1, \infty)$.

Now we find $f'(0) = -6(0+2)(0+1) = -12$ which is negative and so in the region $x \in (-1, \infty)$. (which contains $x = 0$), the function is strictly decreasing.

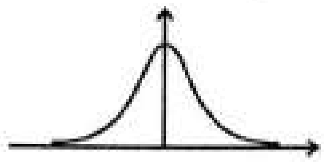
Therefore in the next adjacent region on the left i.e. $x \in (-2, -1)$, the function is strictly increasing and in the next adjacent region on the left $x \in (-\infty, -2)$, the function is again strictly decreasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.



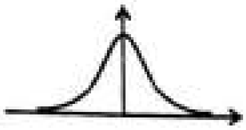
So the final region in which the function strictly increasing is $x \in (-2, -1)$ and the region in which the function is strictly decreasing is $x \in (-\infty, -2) \cup (-1, \infty)$.

ILLUSTRATIVE EXAMPLES FROM GATE

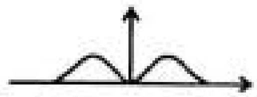
Q.44 The derivative of the symmetric function drawn in given figure will look like



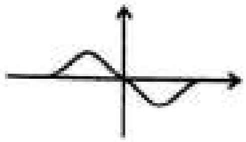
(a)



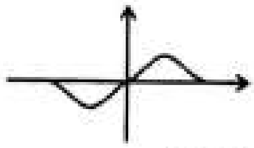
(b)



(c)



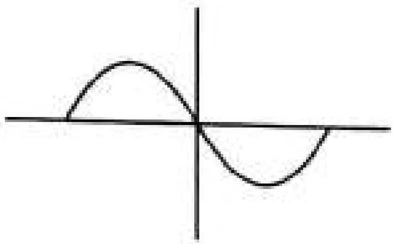
(d)



[EC, GATE-2005, 2 marks]

Solution: (c)

Given function has negative slope in +ve half and +ve slope in –ve half. So its differentiation curve is satisfied by (c).



Q.45 As x increased from $-\infty$ to ∞, the function $f(x) = \frac{e^x}{1+e^x}$

(a) monotonically increases
(b) monotonically decreases
(c) increases to a maximum value and then decreases
(d) decreases to a minimum value and then increases

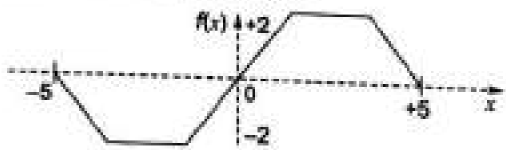
[EC, GATE-2006, 2 marks]

Solution: (a)

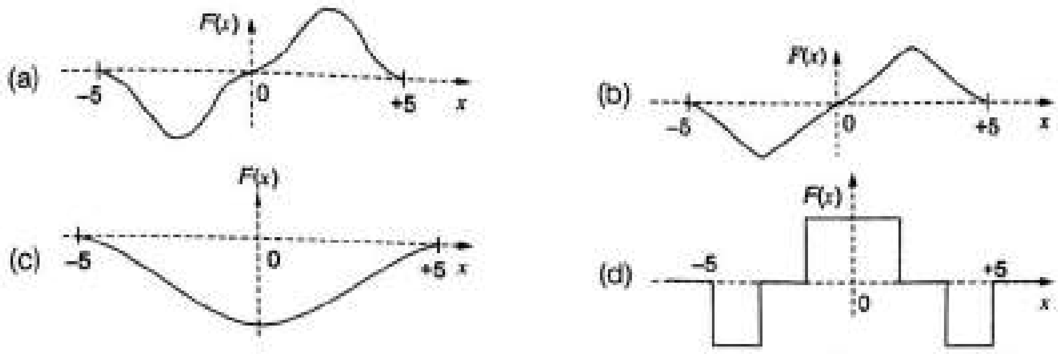
$$f'(x) = \frac{e^x(1+e^x)-e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

since e^x is +ve for all values of x, f'(x) is +ve for all values of x and hence f(x) monotonically increases.

Q.46 Consider the plot $f(x)$ versus x as shown below.



Suppose $F(x) = \int_{-5}^{x} f(y)dy$. Which one of the following is a graph of $F(x)$?



[EC, 2016 : 1 Mark, Set-1]

Solution: (c)

$$F'(x) = f(x) \text{ which is density function}$$
$$F'(x) = f(x) < 0 \text{ when } x < 0$$

$\therefore$ $F(x)$ is decreasing for $x < 0$

$$F'(x) = f(x) > 0 \text{ when } x > 0$$

$\therefore$ $F(x)$ is increasing for $x > 0$

Q.47 Let $f(x)$ be a polynomial and $g(x) = f'(x)$ be its derivative. If the degree of $(f(x) + f(-x))$ is 10, then the degree of $(g(x) - g(-x))$ is ____. [CS, 2016 : 1 Mark, Set-2]

Solution:

If $f(x) + f(-x)$ is degree 10

$f(x) = a_{10}x^{10} + a_9x^9 .... + a_1x + a_0$

$f(-x) = a_{10}x^{10} - a_9x^9 ... - a_1x + a_0$

$f(x) + f(-x) = a_{10}x^{10} + a_8x^8 + ... + a_0$

Now $g(x) = f'(x) = 10a_{10}x^9 + 9a_9x^8 + ... + a_1$

$g(-x) = f'(-x) = -10a_{10}x^9 + 9a_9x^8 + ... + a_1$

$g(x) - g(-x) = 20a_{10}x^9 + ...$

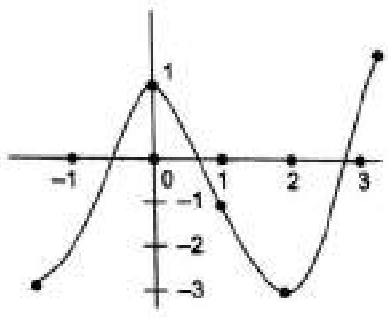
Clearly degree of $(g(x) - g(-x))$ is 9.

Q.48 As x varies from –1 to +3, which one of the following describes the behaviour of the function $f(x) = x^3 - 3x^2 + 1$?

(a) $f(x)$ increases monotonically.
(b) $f(x)$ increases, then decreases and increases again.
(c) $f(x)$ decreases, then increases and decreases again.
(d) $f(x)$ increases and then decreases [EC, 2016 : 1 Mark, Set-2]

Solution: (b)

$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x$$
$$f'(x) = 0$$
$$3x^2 - 6x = 0$$
$$3x(x - 2) = 0$$
$$x = 0, 2$$
$$f''(x) = 6x - 6$$



At $x = 0$ $f''(0) = -6$ maxima

$x = 2$ $f''(2) = 6$ minima

2.6.2 Relative or Local Maxima and Minima (of function of a single independent variable)

Definitions: A function f(x) is said to be a local or relative maximum at x = a, if there exist a positive number δ such that f(a + δ) < f(a) for all values of δ other than zero, in the interval (–δ, δ).

A function f(x) is said to be a local or relative minimum at x = a, if there exists a positive number δ such that f(a + δ) > f(a) for all values of δ, other then zero, in the interval (–δ, δ).

Maximum and Minimum values of a function are together also called extreme values or turning values and the points at which they are attained are called points of maxima and minima.

The points at which a function has extreme values are called Turning Points.

2.6.2.1 Properties of Relative Maxima and Minima

1. At least one maximum or one minimum must lie between two equal values of a function.
2. Maximum and minimum values must occur alternatively.
3. There may be several maximum or minimum values of same function.
4. A function y = f(x) is maximum at x = a, if dy/dx changes sign from +ve to –ve as x passes through a.
5. A function y = f(x) is minimum at x = a, if dy/dx changes sign from –ve and +ve as x passes through a.
6. If the sign of dy/dx does not change while x passes through a, then y is neither maximum nor minimum at x = a.

2.6.2.2 Conditions for Maximum or Minimum Values

The necessary condition that f(x) should have a maximum or a minimum at x = a is that f′(a) = 0.

2.6.2.3 Definition of Stationary Values

A function f(x) is said to be stationary at x = a if f′(a) = 0.

Thus for a function f(x) to be a maximum or minimum at x = a it must be stationary at x = a.

2.6.2.4 Sufficient Conditions of Maximum or Minimum Values

There is a maximum of f(x) at x = a if f′(a) = 0 and f″(a) is negative.

Similarly there is a minimum of f(x) at x = a if f′(a) = 0 and f″(a) is positive.

Note: If f″(a) is also equal to zero, then we can show that for a maximum or a minimum of f(x) at x = a, we must have f‴(a) = 0. Then, if $f^{iv}(a)$ is negative, there will be a maximum at x = a and if $f^{iv}(a)$ is positive there will be minimum at x = a.

In general if, f′(a) = f″(a) = f‴(a) = ... $f^{n-1}(a)$ = 0 and $f^n(a) \neq 0$ then n must an even integer for maximum or minimum. Also for a maximum $f^n(a)$ must be negative and for a minimum $f^n(a)$ must be positive.

2.6.2.5 Working rule for Maxima and Minima of f(x)

1. Find f′(x) and equate to zero.
2. Solve the resulting equation for x. Let its roots be a_1, a_2, ... Then f(x) is stationary at x = a_1, a_2, ... Thus x = a_1, a_2, ... are the only points at which f(x) can be maximum or a minimum.
3. Find f″(x) and substitute in it by terms x = a_1, a_2, ...wherever f″(x) is x we have a maximum and wherever f″(x) is +ve, we have a minimum.
4. If f″(a_1) = 0, find f‴(x) put x = a_1 in it. If f‴(a_1) ≠ 0, there is neither a maximum nor a minimum at x = a_1. If f‴(a_1) = 0, find $f^{iv}(x)$ and put x = a_1 in it. If $f^{iv}(a_1)$ is –ve, we have maximum at x = a_1, if it is positive there is a minimum at x = a_1. If $f^{iv}(a_1)$ is zero, we must find $f^v(x)$, and so on. Repeat the above process for each root of the equation f′(x) = 0.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.49 Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$.
The optimal value of $f(x, y)$
(a) is a minimum equal to 10/3
(b) is a maximum equal to 10/3
(c) is a minimum equal to 8/3
(d) is a maximum equal to 8/3

[CE, GATE-2010, 2 marks]

Solution: (a)

$$f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$$

$$\frac{\partial f}{\partial x} = 8x - 8$$

$$\frac{\partial f}{\partial y} = 12y - 4$$

Putting, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$8x - 8 = 0 \text{ and } 12y - 4 = 0$$

Given, $x = 1$ and $y = \frac{1}{3}$

$\left(1, \frac{1}{3}\right)$ is the only stationary point.

$$r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8$$

$$s = \left[\frac{\partial^2 f}{\partial x dy}\right]_{\left(1, \frac{1}{3}\right)} = 0$$

$$t = \left[\frac{\partial^2 f}{\partial y^2}\right]_{\left(1, \frac{1}{3}\right)} = 12$$

Since, $rt = 8 \times 12 = 96$

$s^2 = 0$

Since, $rt > s^2$,

we have either a maxima or minima at $\left(1, \frac{1}{3}\right)$

also since, $r = \left[\frac{\partial f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8 > 0$, the point $\left(1, \frac{1}{3}\right)$ is a point of minima.

The minimum value is

$$f\left(1, \frac{1}{3}\right) = 4 \times 1^2 + 6 \times \frac{1}{3^2} - 8 \times 1 - 4 \times \frac{1}{3} + 8 = \frac{10}{3}$$

So the optimal value of $f(x, y)$ is a minimum equal to $\frac{10}{3}$.

Q.50 While minimizing the function $f(x)$, necessary and sufficient conditions for a point x_0 to be a minima are

(a) $f'(x_0) > 0$ and $f''(x_0) = 0$
(b) $f'(x_0) < 0$ and $f''(x_0) = 0$
(c) $f'(x_0) = 0$ and $f''(x_0) < 0$
(d) $f'(x_0) = 0$ and $f''(x_0) > 0$

[CE, GATE-2015 : 1 Mark, Set-II]

Solution: (d)

$f(x)$ has a local minimum at $x = x_0$

if $f'(x_0) = 0$

and $f''(x_0) > 0$

Q.51 The distance between the origin and the point nearest to it on the surface $z^2 = 1 + xy$ is

(a) 1
(b) $\frac{\sqrt{3}}{2}$
(c) $\sqrt{3}$
(d) 2

[ME, GATE-2009, 2 marks]

Solution: (a)

Let the point be (x, y, z) on surface $z^2 = 1 + xy$

$$\text{Distance from origin} = l = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$$

$$l = \sqrt{x^2+y^2+1+xy} \qquad [\text{since } z^2 = 1 + xy \text{ is given}]$$

This distance is shortest when l is minimum we need to find minima of $x^2 + y^2 + 1 + xy$

Let $u = x^2 + y^2 + 1 + xy$

$$\frac{\partial u}{\partial x} = 2x + y$$

$$\frac{\partial u}{\partial y} = 2y + x$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

$\Rightarrow$ $2x + y = 0$ and $2y + x = 0$

Solving simultaneously, we get

$x = 0$ and $y = 0$

is the only solution and so (0, 0) is the only stationary point.

Now,
$$r = \frac{\partial^2 u}{\partial x^2} = 2$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2$$

Since $rt = 2 \times 2 = 4 > s^2 = 1$

We have case 1, i.e. either a maximum or minimum exists at (0, 0)

Now, since $r = 2 > 0$, so it is a minima at (0, 0).

Now at $x = 0, \quad y = 0, \quad z = \sqrt{1+xy} = \sqrt{1+0} = 1$

So, the point nearest to the origin on surface $z^2 = 1 + xy$ is (0, 0, 1)

The distance $l = \sqrt{0^2+0^2+1^2} = 1$

So, correct answer is choice (a).

Q.52 At $x = 0$, the function $f(x) = |x|$ has

(a) a minimum
(b) a maximum
(c) a point of inflection
(d) neither a maximum nor minimum

[ME, GATE-2015 : 1 Mark, Set-2]

Answer: (a)

Q.53 The function $f(x) = 2x - x^2 + 3$ has

(a) a maxima at x = 1 and a minima at x = 5
(b) a maxima at x = 1 and a minima at x = –5
(c) only a maxima at x = 1
(d) only a minima at x = 1

[EE, GATE-2011, 2 marks]

Solution: (c)

$$f(x) = 2x - x^2 + 3$$
$$f'(x) = 2 - 2x = 0$$
$$\Rightarrow \quad x = 1 \text{ is the stationary point}$$
$$f''(x) = -2$$
$$\Rightarrow \quad f''(1) = -2 < 0$$

So at x = 1 we have a relative maxima.

Q.54 If the sum of the diagonal elements of a 2 × 2 symmetric matrix is –6, then the maximum possible value of determinant of the matrix is _____.

[EE, GATE-2015 : 1 Mark, Set-1]

Solution: (9)

Consider a symmetric matrix $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

Given $a + d = -6$

$$|A| = ad - b^2$$

Now since b^2 is always non-negative, maximum determinant will come when $b^2 = 0$.

So we need to maximize

$$|A| = ad - 0 = ad = a \times -(6 + a) = -a^2 - 6a$$

$$\frac{d|A|}{da} = -2a - 6 = 0$$

$$\Rightarrow \quad a = -3 \text{ is the only stationary point}$$

Since $\left[\frac{d^2|A|}{da}\right]_{a=-3} = -2 < 0$, we have a maximum at $a = -3$.

Since $a + d = -6$, Corresponding value of $d = -3$.

Now the maximum value of determinant is

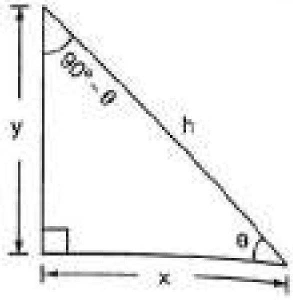
$$|A| = ad = -3 \times -3 = 9$$

Q.55 For a right angled triangle, if the sum of the lengths of the hypotenuse and a side is kept constant, in order to have maximum area of the triangle, the angle between the hypotenuse and the side is

(a) 12°
(b) 36°
(c) 60°
(d) 45°

[EC, GATE-2014 : 2 Marks, Set-4]

Solution : (c)



$$h = \sqrt{x^2 + y^2}$$

Given that,
$$x + \sqrt{x^2 + y^2} = k\,(\text{constant})$$
$$x^2 + y^2 = (k - x)^2$$
$$y^2 = k^2 - 2kx$$

Area,
$$A = \frac{1}{2}\cdot x \cdot y$$
$$A^2 = \frac{x^2}{4}(k^2 - 2kx)$$

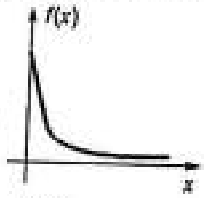
Let,
$$f(x) = A^2 = \frac{x^2}{4}(k^2 - 2kx)$$
$$f'(x) = \frac{1}{4}(2k^2x - 6kx^2)$$
$$f'(x) = 0$$
$$2k^2x - 6kx^2 = 0$$
$$x = \frac{k}{3},\ 0$$

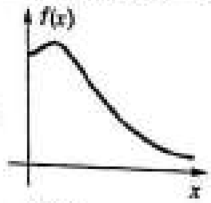
At
$$x = \frac{k}{3},\ f''(x) < 0$$

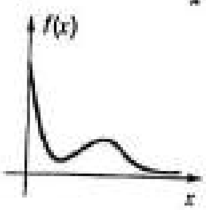
∴ Area is maximum at $x = \frac{k}{3}$

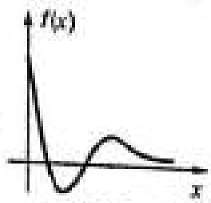
∴
$$y^2 = k^2 - \frac{2k^2}{3} = \frac{k^2}{3}$$
$$y = \frac{k}{\sqrt{3}}$$
$$\tan\theta = \frac{y}{x} = \sqrt{3}$$
$$\theta = 60^\circ$$

Q.56 Which one of the following graphs describes the function $f(x) = e^{-x}(x^2 + x + 1)$?

(a)


(b)


(c)


(d)


[EC, GATE-2015 : 2 Marks, Set-1]

Solution: (b)

$$f(x) = e^{-x}(x^2 + x + 1)$$
$$f'(x) = e^{-x}(2x + 1) - e^{-x}(x^2 + x + 1)$$
$$= e^{-x}(x - x^2) = e^{-x}(x)(1 - x)$$

Putting $f'(x) = 0$, we get

$$x = 0 \quad \text{or} \quad x = 1$$
$$f''(x) = e^{-x}(1 - 2x) - e^{-x}(x - x^2) = e^{-x}(1 - 3x + x^2)$$

At $x = 0$, $f''(x) = 1$ (so we have a minimum).

At $x = 1$, $f''(x) = -\frac{1}{e}$ (so we have a maximum).

Only curve (b) shows a single local minimum at $x = 0$ and a single local maximum at $x = 1$.

Q.57 The maximum area (in square unit) of a rectangle whose vertices lies on the ellipse $x^2 + 4y^2 = 1$ is_______.

[EC, GATE-2015 : 2 Marks, Set-1]

Solution: (4)

$$x^2 + 4y^2 = 1$$

Area of rectangle

$$= 2x \cdot 2y = 4xy$$

Let $f = (\text{Area})^2 = 16x^2y^2$

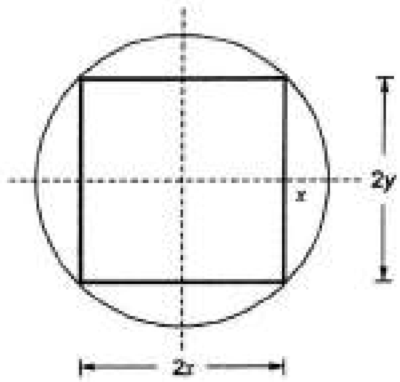
$4x^2(1 - x^2)$ $\quad (\because 1 - x^2 = 4y^2)$

$$f'(x) = 0$$

We get, $x = \frac{1}{\sqrt{2}}$

$$y = \frac{1}{\sqrt{8}}$$

$$\text{Area} = 4xy = 4 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{8}} = 1$$



2.6.3 Working Rules for Finding (Absolute) Maximum and Minimum in Range [a, b]

If a function f is differentiable in [a, b] except (possibly) at finitely many points, then to find (absolute) maximum and minimum values adopt the following procedure:

1. Evaluate f(x) at the points where f'(x) = 0.
2. Evaluate f(x) at the points where derivative fails to exist.
3. Find f(a) and f(b).

Then the maximum of these values is the absolute maximum of the given function f and the minimum of these values is the absolute minimum of the given function f.

ILLUSTRATIVE EXAMPLES

Example: 1

Find the absolute maximum and minimum values of:

(a) $f(x) = 2x^3 - 9x^2 + 12x - 5$ in [0, 3]

(b) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

Also find points of maxima and minima.

Solution:

(a) Given $f(x) = 2x^3 - 9x^2 + 12x - 5$ ... (i)

It is differentiable for all x in [0, 3], since it is a polynomial

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2.3x^2 - 9.2x + 12 = 6(x^2 - 3x + 2)$$

Now, $f'(x) = 0$

$\Rightarrow$ $6(x^2 - 3x + 2) = 0$

$\Rightarrow$ $x^2 - 3x + 2 = 0$

$\Rightarrow$ $(x - 1)(x - 2) = 0$

$\Rightarrow$ $x = 1, 2$

Also 1, 2 both are in [0, 3], therefore 1 and 2 both are stationary points or turning points.

Further, $f(1) = 2.1^3 - 9.1^2 + 12.1 - 5 = 2 - 9 + 12 - 5 = 0$

$f(2) = 2.2^3 - 9.2^2 + 12.2 - 5 = 16 - 36 + 24 - 5 = -1$

$f(0) = -5$

and $f(3) = 2.3^3 - 9.3^3 + 12.3 - 5 = 54 - 81 + 36 - 5 = 4$

Therefore, the absolute maximum value = 4 and the absolute minimum value = − 5. The point of maxima is 3 and the point of minima is 0.

(b) Given, $f(x) = 12x^{4/3} - 6x^{1/3}, x \in [-1, 1]$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 12.\frac{4}{3}x^{1/3} - 6.\frac{1}{3}x^{-2/3} = 16x^{1/3} - \frac{2}{x^{2/3}} = \frac{2(8x-1)}{x^{2/3}}$$

Now, $f'(x) = 0$

$\Rightarrow$ $\frac{2(8x-1)}{x^{2/3}} = 0$

$\Rightarrow$ $x = \frac{1}{8}$

As $\frac{1}{8} \in [-1, 1]$, $\frac{1}{8}$ is a critical point.

Also we note that f is not differentiable at x = 0.

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{4/3} - 6\left(\frac{1}{8}\right)^{1/3} = 12\left(\frac{1}{2}\right)^4 - 6.\frac{1}{2}$$

$$= 12.\frac{1}{16} - 3 = \frac{3}{4} - 3 = -\frac{9}{4}$$

$$f(0) = 12.0 - 6.0 = 0$$

$$f(-1) = 12(-1)^{4/3} - 6(-1)^{1/3} = 12.1 - 6.(-1) = 18$$

$$f(1) = 12.1^{4/3} - 6/1^{1/3} = 12.1 - 6.1 = 6$$

Therefore, the absolute maximum value = 18 and the absolute minimum value = $-\frac{9}{4}$. The point of maxima is −1 and the point of minima is $\frac{1}{8}$.

Example: 2

It is given that at x = 1, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value in the interval [0, 2]. Find the value of a.

Solution:

Let $f(x) = x^4 - 62x^2 + ax + 9$ ... (i)

It is differentiable for all x in [0, 2].

Differentiating (i) w.r.t. x, we get

$$f'(x) = 4x^3 - 124x + a$$

$$\therefore \quad f'(1) = 4.1^3 - 124.1 + a = a - 120$$

Given that at x = 1, the function (i) has maximum value, therefore, x = 1 is a point of maxima

$\Rightarrow$ x = 1 is a critical point

$\Rightarrow \quad f'(1) = 0$

$\Rightarrow \quad a - 120 = 0$

$\Rightarrow \quad a = 120$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.58 The function $f(x) = 2x^3 - 3x^2 - 36x + 2$ has its maxima at

(a) $x = -2$ only
(b) $x = 0$ only
(c) $x = 3$ only
(d) both $x = -2$ and $x = 3$

[CE, GATE-2004, 2 marks]

Solution: (a)

Putting

$$f'(x) = 6x^2 - 6x - 36 = 0$$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad x = 3 \text{ or } -2$

Now $\quad f''(x) = 12x - 6$

and $\quad f''(3) = 30 > 0$ (minima)

and $\quad f''(-2) = -30 < 0$ (maxima)

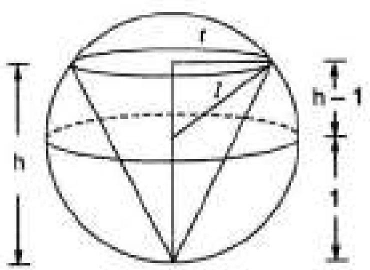
Hence maxima is at $x = -2$ only.

Q.59 The right circular cone of largest volume that can be enclosed by a sphere of 1 m radius has a height of

(a) 1/3 m
(b) 2/3 m
(c) $\frac{2\sqrt{2}}{3}$ m
(d) 4/3 m

[ME, GATE-2005, 2 marks]

Solution: (d)



$$r^2 + (h-1)^2 = 1^2$$

$$r^2 + h^2 - 2h + 1 = 1$$

$$r^2 = 2h - h^2$$

Volume of the cone, $$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{\pi}{3}(2h - h^2)h = \frac{\pi}{3}(2h^2 - h^3)$$

$$\frac{dV}{dh} = \frac{\pi}{3}(4h - 3h^2)$$

$$\frac{dV}{dh} = 0 \qquad \text{for minima and maxima}$$
$$4h - 3h^2 = 0$$
$$h(4-3h) = 0$$
$$h = \frac{4}{3}, 0$$
$$V'' = \frac{\pi}{3}(4-6h)$$
$$h = 0 \,;\, V'' = \frac{4\pi}{3} > 0 \text{ minima}$$
$$h = \frac{4}{3} \,;\, V'' = -\frac{4\pi}{3} < 0 \text{ maxima}$$

$\therefore$ Volume is maximum when $x = \frac{4}{3}$

Q.60 The minimum value of function $y = x^2$ in the interval [1, 5] is

(a) 0
(b) 1
(c) 25
(d) undefined

[ME, GATE-2007, 1 mark]

Solution: (b)

Given, $y = x^2$

$\Rightarrow \quad \frac{dy}{dx} = 2x = 0$ at $x = 0$

$\Rightarrow \quad \left.\frac{d^2y}{dx^2}\right]_{x=0} = 2$ which is +ve

so we have a local minima at $x = 0$

at $x = 0, \quad y = 0$

but since $x = 0 \notin [1, 5]$

it is not a candidate for minima or maxima in that range

At the end point $x = 1$

$y = 1$

at second end point $x = 5$

$y = 25$

So, absolute minimum value of function in [1, 5] is 1.

Q.61 At $x = 0$, the function $f(x) = x^3 + 1$ has

(a) a maximum value
(b) a minimum value
(c) a singularity
(d) a point of inflection

[ME, GATE-2012, 1 mark]

Solution: (d)

$f(x) = x^3 + 1$

Put $f'(x) = 0$

$\Rightarrow \quad 3x^2 = 0$

$\Rightarrow \quad x = 0$ is the only critical point at this critical point

$$f''(x) = 6x$$
$$f''(0) = 6 \times 0 = 0$$

Now $\quad f'''(x) = 6$ and

so $\quad f'''(0) = 6$ which is non zero.

Since the first non zero derivative value occurs at the third derivative which is an odd derivative, this function has a point of inflection at $x = 0$.

Q.62 For the function $f(x) = x^2 e^{-x}$, the maximum occurs when x is equal to

(a) 2 (b) 1
(c) 0 (d) –1 **[EE, GATE-2005, 2 marks]**

Solution: (a)

$$f(x) = x^2 e^{-x}$$
$$f'(x) = x^2(-e^{-x}) + e^{-x} \times 2x = e^{-x}(2x - x^2)$$

Putting $f'(x) = 0$

$$e^{-x}(2x - x^2) = 0$$
$$e^{-x} x(2 - x) = 0$$

$x = 0$ or $x = 2$ are the stationary points.

Now, $\quad f''(x) = e^{-x}(2 - 2x) + (2x - x^2)(-e^{-x})$
$$= e^{-x}(2 - 2x - (2x - x^2)) = e^{-x}(x^2 - 4x + 2)$$

at $x = 0$, $\quad f''(0) = e^{-0}(0 - 0 + 2) = 2$

Since $f''(x) = 2$ is > 0 at $x = 0$ we have a minima.

Now at $x = 2$ $\quad f''(2) = e^{-2}(2^2 - 4 \times 2 + 2)$
$$= e^{-2}(4 - 8 + 2)$$
$$= -2e^{-2} < 0$$

$\therefore$ at $x = 2$ we have a maxima.

Q.63 Consider function $f(x) = (x^2 - 4)^2$ where x is a real number. Then the function has

(a) only one minimum (b) only two minima
(c) three minima (d) three maxima

[EE, GATE-2008, 2 marks]

Solution: (b)

$$f(x) = (x^2 - 4)^2$$
$$f'(x) = 2(x^2 - 4) \times 2x = 4x(x^2 - 4) = 0$$

$x = 0$, $x = 2$ and $x = -2$ are the stationary pts.

$$f''(x) = 4[x(2x) + (x^2 - 4) \times 1]$$
$$= 4[2x^2 + x^2 - 4] = 4[3x^2 - 4] = 12x^2 - 16$$

$f''(0) = -16 < 0$ (so maxima at $x = 0$)

$f''(2) = (12)2^2 - 16 = 32 > 0$ (so minima at $x = 2$)

$f''(-2) = 12(-2)^2 - 16 = 32 > 0$ (so minima at $x = -2$)

$\therefore$ There is only one maxima and only two minima for this function.

Q.64 A cubic polynomial with real coefficients

(a) can possibly have no extrema and no zero crossings
(b) may have up to three extrema and upto 2 zero crossings
(c) cannot have more than two extrema and more than three zero crossings
(d) will always have an equal number of extrema and zero crossings

[EE, GATE-2009, 2 marks]

Solution: (c)

An n^{th} degree polynomial bends exactly n − 1 times and therefore can have a maximum of n − 1 extremas. Also an n^{th} degree polynomial has at most n roots (zero crossings). So a cubic polynomial (degree 3) cannot have more than 2 extrema and cannot have more than 3 zero crossings.

Q.65 At t = 0, the function $f(t) = \frac{\sin t}{t}$ has

(a) a minimum
(b) a discontinuity
(c) a point of inflection
(d) a maximum

[EE, GATE-2010, 2 marks]

Solution: (d)

$$f(t) = \frac{\sin t}{t}$$

$$f(t) = \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots}{t}$$

$$f(t) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots$$

$$f'(t) = -\frac{2t}{3!} + \frac{4t^3}{5!} - \cdots$$

$$f''(t) = -\frac{2t}{3!} + \frac{4t^3}{5!} - \cdots$$

At $t = 0$, $f'(t) = 0$, $f''(t) < 0$

∴ f(t) attains maxima.

Q.66 A function $y = 5x^2 + 10x$ is defined over an open interval x = (1, 2). At least at one point in this interval, $\frac{dy}{dx}$ is exactly

(a) 20
(b) 25
(c) 30
(d) 35

[EE, GATE-2013, 2 Marks]

Solution: (b)

$$\frac{dy}{dx} = 10x + 10$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 20$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 30$$

∵ x is defined open interval x = (1, 2)

∴ $1 < x < 2$

∴ $20 < \frac{dy}{dx} < 30$

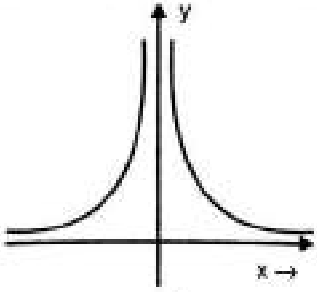
Q.67 Which one of the following functions is strictly bounded?

(a) $1/x^2$ (b) e^x

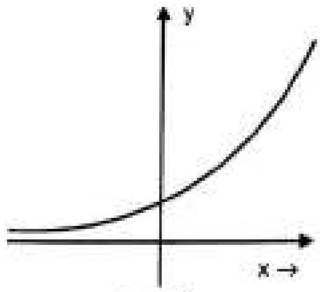
(c) x^2 (d) e^{-x^2} **[EC, GATE-2007, 1 mark]**

Solution: (d)

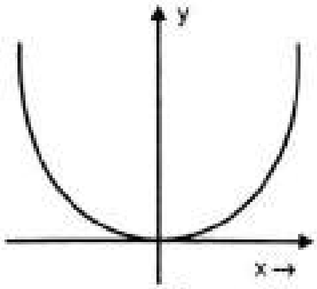
From the graphs below, we can see that only e^{-x^2} is strictly bounded



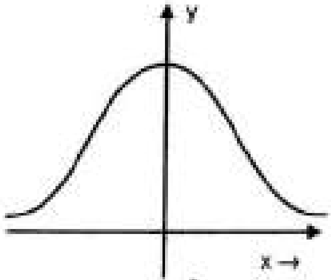
$y = \frac{1}{x^2}$



$y = e^x$



$y = x^2$



$y = e^{-x^2}$

Q.68 Consider the function $f(x) = x^2 - x - 2$. The maximum value of $f(x)$ in the closed interval $[-4, 4]$ is

(a) 18 (b) 10

(c) −2.25 (d) indeterminate

[EC, GATE-2007, 2 marks]

Solution: (a)

$$f(x) = x^2 - x - 2 = (x+1)(x-2)$$

$$f'(x) = 2x - 1 = 0$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$f''(x) = 2$$

$$f''\left(\frac{1}{2}\right) = 2 > 0$$

So at $x = \frac{1}{2}$,

we have a local minima so this is not a candidate for maxima in range $[-4, 4]$.

Now $f(-4) = 18$

$f(+4) = 10$

so maximum value in range $[-4, 4]$ is 18.

Q.69 If $e^y = x^{\frac{1}{x}}$, then y has a

(a) maximum at $x = e$ (b) minimum at $x = e$

(c) maximum at $x = e^{-1}$ (d) minimum at $x = e^{-1}$

[EC, GATE-2010, 2 marks]

Solution: (a)

$$e^y = x^{1/x}$$

Taking log on both sides,

$$y = \frac{1}{x}\log x$$

$$\frac{dy}{dx} = \frac{1}{x}\cdot\frac{1}{x} + \log x\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}(1-\log x)$$

putting $\frac{dy}{dx} = 0$

$$\frac{1}{x^2}(1-\log x) = 0$$

$\Rightarrow$ $\log x = 1$

$\Rightarrow$ $x = e$ is a stationary point

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\times\left(-\frac{1}{x}\right) + (1-\log x)\times\left(-\frac{2}{x^3}\right)$$

$$= -\frac{1}{x^3}[1+2(1-\log x)] = -\frac{1}{x^3}(3-2\log x)$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=e} = -\frac{1}{e^3}(3-2\log e) = -\frac{1}{e^3}$$

which is negative.

So, at $x = e$, we have a maximum.

Q.70 A point on a curve is said to be an extremum if it is a local minimum or a local maximum. The number of distinct exterma for the curve $3x^4 - 16x^3 - 24x^2 + 37$ is

(a) 0 (b) 1

(c) 2 (d) 3

[CS, GATE-2008, 2 marks]

Solution: (d)

$$y = 3x^4 - 16x^3 - 24x^2 + 37$$

$$\frac{dy}{dx} = 12x^3 - 48x^2 - 48x = 0$$

$$x(12x^2 - 48x - 48) = 0$$

$$x = 0$$

or

$$12x^2 - 48x - 48 = 0$$

$$x^2 - 4x - 4 = 0$$

$$x = \frac{4 \pm \sqrt{16+16}}{2} = \frac{4\pm\sqrt{32}}{2} = \frac{4\pm 4\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

$$\frac{d^2y}{dx^2} = 36x^2 - 96x - 48$$

Now at $x = 0$

$$\frac{d^2y}{dx^2} = -48 \neq 0$$

at $1 \pm \sqrt{2}$ also $\frac{d^2y}{dx^2} \neq 0$ (using calculator)

$\therefore$ There are 3 extrema in this function.

Q.71 The maximum value of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is

(a) 21 (b) 25

(c) 41 (d) 46

[EC, EE, IN, GATE-2012, 2 marks]

Solution: (c)

We need absolute maximum of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6]

First find local maximum if any by putting $f'(x) = 0$.

i.e. $f'(x) = 3x^2 - 18x + 24 = 0$

i.e. $x^2 - 6x + 8 = 0$

$x = 2, 4$

Now $f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -6 < 0$ (So x = 2 is a point of local maximum)

and $f''(4) = 24 - 18 = +6 > 0$ (So x = 4 is a point of local minimum)

Now tabulate the values of f at end point of interval and at local maximum point, to find absolute maximum in given range, as shown below:

x	f(x)
1	21
2	25
6	41

Clearly the absolute maxima is at x = 6

and absolute maximum value is 41.

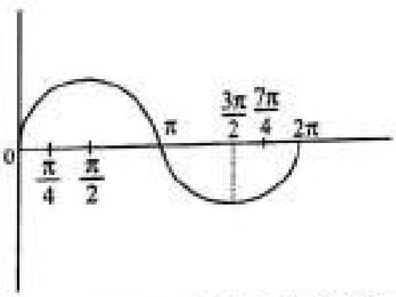
Q.72 Consider the function f(x) = sin (x) in the interval $x \in [\pi/4, 7\pi/4]$. The number and location(s) of the local minima of this function are

(a) One, at $\pi/2$ (b) One, at $3\pi/2$

(c) Two, at $\pi/2$ and $3\pi/2$ (d) Two, at $\pi/4$ and $3\pi/2$

[CS, GATE-2012, 1 mark]

Solution: (b)



From the plot of sin x given above, we can easily see that in the range $[\pi/4, 7\pi/4]$, there is only one local minima, at $3\pi/2$.

Q.73 Let $f: [-1, 1] \rightarrow R$, where $f(x) = 2x^3 - x^4 - 10$. The minimum value of $f(x)$ is______.

[IN, 2016 : 2 Marks]

Solution:

$$f(x) = 2x^3 - x^4 - 10 \quad \text{in } [-1, 1]$$

$$f'(x) = 6x^2 - 4x^3$$

for minima and maxima

$$f'(x) = 0$$

$$6x^2 - 4x^3 = 0$$

$$2x^2(3 - 2x) = 0$$

$$x = 0, 0, \frac{3}{2}$$

$$f''(x) = 12x - 12x^2 \text{ s}$$

for $x = 0$ $\quad f''(0) = 0$

for $x = \frac{3}{2}$ $\quad f''\left(\frac{3}{2}\right) = 18 - 27 = -9 < 0$ maxima

at $\quad x = -1$ $f(-1) = -2 - 1 - 10 = -13$

at $\quad x = 1$ $f(1) = 2 - 1 - 10 = -9$

At $x = -1$, function attains global minimum value with $f(x)_{min} = -13$.

Q.74 The maximum value attained by the function $f(x) = x(x - 1)(x - 2)$ in the interval [1, 2] is ______

[EE, 2016 : 1 Mark, Set-1]

Solution:

$$f(x) = x^3 - 3x^2 + 2x \quad [1, 2]$$

$$f'(x) = 3x^2 - 6x + 2$$

$f'(x) = 0$ for stationary point

stationary points are $1 \pm \frac{1}{\sqrt{3}}$

only $1 + \frac{1}{\sqrt{3}}$ lies in [1, 2]

$$f(1) = 0$$

$$f(2) = 0$$

$$f\left(1 + \frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$$

Maximum value is 0.

Q.75 The optimum value of the function $f(x) = x^2 - 4x + 2$ is

(a) 2 (maximum)
(b) 2 (minimum)
(c) −2 (maximum)
(d) −2 (minimum)

[CE, 2016 : 1 Mark, Set-II]

Solution: (d)

$$f'(x) = 0$$

$\Rightarrow$ $\quad 2x - 4 = 0$

$\Rightarrow$ $\quad x = 2$ (stationary point)

$$f''(x) = 2 > 0$$

$\Rightarrow$ $f(x)$ is minimum at $x = 2$

i.e., $(2)^2 - 4(2) + 2 = -2$

$\therefore$ The optimum value of $f(x)$ is −2 (minimum)

2.6.4 Taylor's and Maclaurin's Series Expansion of Functions

2.6.4.1 Taylor's Series

If (i) f(x) and its first (n − 1) derivatives be continuous in [a, a + h], and (ii) $f^n(x)$ exists for every value of x in (a, a + h), then there is at least one number θ(0 < θ < 1), such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a+\theta h) \quad \ldots (i)$$

which is called Taylor's theorem with Lagrange's form of remainder, the remainder R_n being $\frac{h^n}{n!}f^n(a+\theta h)$.

Consider the function $$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots + \frac{(a+h-x)^n}{n!}K$$

where K is defined by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}K \quad \ldots (ii)$$

1. Since f(x), f'(x), ..., f^{n-1} (x) are continuous in [a, a + h], therefore φ(x) is also continuous in [a, a + h].
2. φ'(x) exists and = $\frac{(a+h-x)^{n-1}}{(n-1)!}[f^n(x) - K]$
3. Also $\phi(a) = \phi(a+h)$ [By (ii)]

Hence φ(x) satisfies all the conditions of Rolle's theorem, and therefore, there exists at least one number θ(0 < θ < 1), such that φ'(a + θh) = 0 i.e. K = $f^n(a + \theta h)$ (0 < θ < 1)

Substituting this value of K in (2), we get (1).

Cor. 1. Taking n = 1 in(1), Taylor's theorem reduces to Lagrange's Mean-value theorem.

Cor. 2. Putting a = 0 and h = x in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) \quad \ldots (iii)$$

which is known as Maclaurin's theorem with Lagrange's form of remainder.

ILLUSTRATIVE EXAMPLES

Example:

If f(x) = log(1 + x), x > 0, using Taylor's theorem, show that for 0 < θ < 1,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

Solution:

Deduce that $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for x > 0.

By Maclaurin's theorem with remainder R_3, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \quad \ldots (i)$$

Here $f(x) = \log(1+x),\quad f(0) = 0$

∴ $f'(x) = \frac{1}{1+x},\qquad f'(0) = 1$

$$f''(x) = \frac{-1}{(1+x)^2}, \quad f''(0) = -1$$

and $$f'''(x) = \frac{2}{(1+x)^3}, \quad f'''(\theta x) = \frac{2}{(1+\theta x)^3}$$

Substituting in (i), we get $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$ ...(ii)

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

or $(1+\theta x)^3 > 1$ i.e. $\frac{1}{(1+\theta x)^3} < 1$

$$\therefore \quad x - x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence $$\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$ [by (ii)]

2.6.4.2 Maclaurin's Series

If f(x) can be expanded as an infinite series, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\infty \quad \ldots(i)$$

If f(x) possesses derivatives of all orders and the remainder R_n in (3) on page 154 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (1).

ILLUSTRATIVE EXAMPLES

Example:

Using Maclaurin's series, expand tan x upto the term containing x^5.

Solution:

Let
$$f(x) = \tan x \qquad f(0) = 0$$
$$\therefore \quad f'(x) = \sec^2 x = 1 + \tan^2 x \qquad f'(0) = 1$$
$$f''(x) = 2\tan x \sec^2 x = 2\tan x(1+\tan^2 x)$$
$$= 2\tan x + 2\tan^3 x \qquad f''(0) = 0$$
$$f'''(0) = 2\sec^2 x + 6\tan^2 x \sec^2 x$$
$$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$$
$$= 2 + 8\tan^2 x + 6\tan^4 x \qquad f'''(0) = 2$$
$$f''''(0) = 16\tan x\sec^2 x + 24\tan^3 x\sec^2 x$$
$$= 16\tan x(1+\tan^2 x) + 24\tan^3 x(1+\tan^2 x)$$
$$= 16\tan x + 40\tan^3 x + 24\tan^5 x \qquad f''''(0) = 0$$
$$f'''''(0) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x)$$
$$f'''''(0) = 16$$

and so on.

Substituting the values of f(0), f'(0), etc. in the Maclaurin's series, we get

$$\tan x = 0 + x.1 + 1 + \frac{x^2}{2!} + \frac{3^3}{3!}\cdot 2 + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 16 + \ldots\ldots$$
$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$$

2.6.4.3 Expansion by Use of Known Series

When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series

1. $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \ldots\ldots$

2. $\sinh\theta = \theta + \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \dfrac{\theta^7}{7!} + \ldots\ldots$

3. $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \dfrac{\theta^6}{6!} + \ldots\ldots$

4. $\cosh\theta = 1 + \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \dfrac{\theta^6}{6!} + \ldots\ldots$

5. $\tan\theta = \theta + \dfrac{\theta^3}{2} + \dfrac{\theta}{15}\theta^5 + \ldots\ldots$

6. $\tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots\ldots$

7. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots\ldots$

8. $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots\ldots$

9. $\log(1-x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots\ldots\right)$

10. $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$

ILLUSTRATIVE EXAMPLES

Example:

Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.

Solution:

We have,

$$e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \frac{(\sin x)^4}{4!} + \ldots\ldots$$

$$= 1 + \left(x - \frac{x^3}{3!} + \ldots\ldots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \ldots\ldots\right) + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \ldots\ldots\right) + \frac{1}{4!}(x - \ldots)^4 + \ldots$$

$$= 1 + \left(x - \frac{x^3}{6} + \ldots\ldots\right) + \frac{1}{2}\left(x^2 - \frac{x^3}{3} + \ldots\ldots\right) + \frac{1}{6}(x^3 - \ldots) + \frac{1}{24}(x^4 + \ldots) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\ldots$$

Otherwise, let

$$f(x) = e^{\sin x} \qquad f(0) = 1$$

$\therefore$

$$f'(x) = e^{\sin x}\cos x\, f(x)\,.\,\cos x, \qquad f'(0) = 1$$

$$f''(x) = f'(x)\cos x - f(x)\sin x, \qquad f''(0) = 1$$

$$f'''(x) = f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x, \qquad f'''(0) = 0$$

$$f''''(x) = f'''(x)\cos x - 3f''(x)\sin x - 3f'(x)\cos x\, f(x)\sin x, \quad f''''(0) = 0$$

and so on

substituting the values of f(0), f'(0) etc., in the Maclaurin's series, we obtain

$$e^{\sin x} = 1 + x\cdot 1 + \frac{x^2}{2!}\cdot 1 + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}\cdot(-3) + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\ldots$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.76 The infinite series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$ corresponds to

(a) sec x
(b) e^x
(c) cos x
(d) $1 + \sin^2 x$

[CE, GATE-2012, 1 mark]

Solution: (b)

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots \quad \text{(By McLaurin's series expansion)}$$

Q.77 In the Taylor series expansion of e^x about x = 2, the coefficient of $(x-2)^4$ is

(a) 1/4!
(b) $2^4/4!$
(c) $e^2/4!$
(d) $e^4/4!$

[ME, GATE-2008, 1 mark]

Solution: (c)

f(x) in the neighbourhood of a is,

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$$

where,
$$b_n = \frac{f^n(a)}{n!}$$

$$f^4(x) = e^x; \quad f^4(2) = e^2$$

$\therefore$ Coefficient of $(x-2)^4 = b_4 = \frac{f^4(2)}{4!} = \frac{e^2}{4!}$

Q.78 A series expansion for the function sin θ is

(a) $1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots$
(b) $\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots$
(c) $1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\ldots$
(d) $\theta+\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\ldots$

Solution: (b)

[ME, GATE-2011, 1 mark]

$$\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^2}{5!}\cdots$$

Q.79 For the function e^{-x}, the linear approximation around x = 2 is

(a) $(3-x)e^{-2}$
(b) $1-x$
(c) $[3+2\sqrt{2}-(1+\sqrt{2})x]e^{-2}$
(d) e^{-2}

[EC, GATE-2007, 1 mark]

Solution: (a)

The Taylor's series expansion of f(x) allowed x = 2 is

$$f(x) = f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)\ldots$$

For linear approximation we take only the first two terms and get

$$f(x) = f(2)+(x-2)f'(2)$$

Here,
$$f(x) = e^{-x} \text{ and } f'(x) = -e^{-x}$$

$\therefore$
$$f(x) = e^{-2}+(x-2)(-e^{-2}) = (3-x)e^{-2}$$

Q.80 Which of the following functions would have only odd powers of x in its Taylor series expansion about the point x = 0?

(a) $\sin(x^3)$ (b) $\sin(x^2)$

(c) $\cos(x^3)$ (d) $\cos(x^2)$ [EC, GATE-2008, 1 mark]

Solution: (a)

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \ldots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots$$

From this,
$$\sin x^2 = x^2 - \frac{x^6}{\lfloor 3} + \frac{x^{10}}{\lfloor 5} - \frac{x^{14}}{\lfloor 7}$$

$$\cos x^2 = 1 - \frac{x^4}{\lfloor 2} + \frac{x^8}{\lfloor 4} - \frac{x^{12}}{\lfloor 6}$$

So, $\sin x^2$ and $\cos x^2$ have only even powers of x

Similarly,
$$\sin x^3 = x^3 - \frac{x^9}{\lfloor 3} + \frac{x^{15}}{\lfloor 5} \ldots$$

$$\cos x^3 = 1 - \frac{x^6}{\lfloor 2} + \frac{x^{12}}{\lfloor 4} \ldots$$

So, only $\sin(x^3)$ has all odd powers of x.

∴ correct choice is (a).

Q.81 In the Taylor series expansion of exp(x) + sin (x) about the point x = π, the coefficient of $(x - \pi)^2$ is

(a) exp (π) (b) 0.5 exp (π)

(c) exp (π) + 1 (d) exp (π) − 1 [EC, GATE-2008, 2 marks]

Solution: (b)

$$f(x) = e^x + \sin x$$

We wish to expand about $x = \pi$

Taylor's series expansion about x = a is

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) \ldots$$

Now about $x = \pi$

$$f(x) = f(\pi) + (x-\pi) f'(\pi) + \frac{(x-\pi)^2}{2!} f''(\pi) + \ldots$$

The coefficient of $(x-\pi)^2$ is $\frac{f''(\pi)}{2!}$

Here
$$f(x) = e^x + \sin x$$
$$f'(x) = e^x + \cos x$$
$$f''(x) = e^x - \sin x$$
$$f''(\pi) = e^\pi - \sin\pi = e^\pi - 0 = e^\pi$$

The coefficient of $(x-\pi)^2$ is therefore $\frac{e^\pi}{2!} = 0.5\exp(\pi)$

Q.82 The Taylor series expansion of $\frac{\sin x}{x-\pi}$ at $x = \pi$ is given by

(a) $1+\frac{(x-\pi)^2}{3!}+...$
(b) $-1-\frac{(x-\pi)^2}{3!}+...$
(c) $1-\frac{(x-\pi)^2}{3!}+...$
(d) $-1+\frac{(x-\pi)^2}{3!}+...$

[EC, GATE-2009, 2 marks]

Solution: (b)

Let,
$$x-\pi = t$$
$$x = \pi + t$$
$$f(t) = \frac{-\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right)}{t}$$
$$f(t) = -\left(1-\frac{t^2}{3!}+\frac{t^4}{5!}-\cdots\right)$$
$$f(t) = -1+\frac{t^2}{3!}-\frac{t^4}{5!}+\cdots$$
$$= -1+\frac{(x-\pi)^2}{3!}-\frac{(x-\pi)^4}{5!}+\cdots$$

Q.83 The quadratic approximation of
$f(x) = x^3 - 3x^2 - 5$ a the point $x = 0$ is

(a) $3x^2 - 6x - 5$
(b) $-3x^2 - 5$
(c) $-3x^2 + 6x - 5$
(d) $3x^2 - 5$

[CE, 2016 : 2 Marks, Set-II]

Solution: (b)

The quadratic approximation of $f(x)$ at the point $x = 0$ is
$$f(x) = f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0) = (-5)+x\{0\}+\frac{x^2}{2}\{-6\} = -3x^2-5$$

2.6.5 Slope Determination of Line

1. This is used to determine slope of straight line in xy plane. For example $y = x + 3$ is a line its slope is given by $\frac{dy}{dx} = 1$.

2. If two lines are perpendicular then product of their slopes is -1.
For example let m_1 be the slope of first line and m_2 is the slope of second line. If both lines are perpendicular then
$$m_1 \cdot m_2 = -1$$

3. The derivatives are also used to find slope of tangent on any curve.
For example $y = f(x)$ is a curve in x-y plane
$$\frac{dy}{dx} = f'(x)\Big|_{(x_0,\, y_0)} \text{ is the slope the tangent at point } (x_0, y_0)$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.84 A polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x - a_0$ with all coefficients positive has
- (a) no real roots
- (b) no negative real root
- (c) odd number of real roots
- (d) at least one positive and one negative real root

[EC, GATE-2013, 1 Mark]

Solution: (d)

Using R-H criterion

$$\begin{array}{c|ccc} x^4 & a_4 & a_2 & -a_0 \\ x^3 & a_3 & a_1 & \\ x^2 & A & & \\ x^1 & a_1 & & \\ x^0 & -a_0 & & \end{array}$$

$$\text{Where } A = \frac{a_3 a_2 - a_1 a_4}{a_3}$$

So, from the above table it is clear that there is atleast one sign change in the first column. So, at least one positive and one negative real root.

Q.85 The angle of intersection of the curves $x^2 = 4y$ and $y^2 = 4x$ at point (0, 0) is
- (a) 0°
- (b) 30°
- (c) 45°
- (d) 90°

[CE, 2016 : 2 Marks, Set-II]

Solution: (d)

Given curve

$$x^2 = 4y \quad \text{...(i)}$$

and

$$y^2 = 4x \quad \text{...(ii)}$$

$$2x = 4\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = 0 = m_1 \text{ (say)}$$

$$2y\frac{dy}{dx} = 4$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = \infty = m_2$$

Let $m_2 = \frac{1}{m'}$, where $m' = 0$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{m_1 m' - 1}{m' + m_1}\right| = \left|\frac{0 - 1}{0 + 0}\right| = \infty$$

$$\Rightarrow \theta = \frac{\pi}{2} = 90°$$

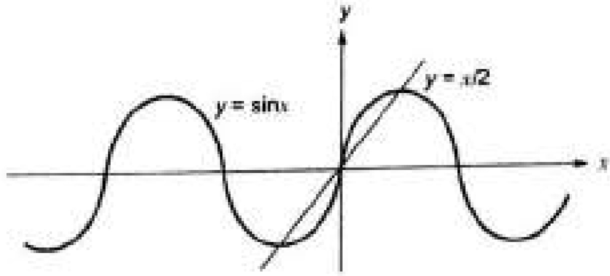
Q.86 How many distinct values of x satisfy the equation $\sin(x) = x/2$, where x is in radians?

(a) 1 (b) 2

(c) 3 (d) 4 or more

[EC, 2016 : 1 Mark, Set-2]

Solution: (c)



Hence 3 solutions.

Q.87 Let $f(x)$ be a polynomial and $g(x) = f'(x)$ be its derivative. If the degree of $(f(x) + f(-x))$ is 10, then the degree of $(g(x) - g(-x))$ is ____.

[CS, 2016 : 1 Mark, Set-2]

Solution:

If $f(x) + f(-x)$ is degree 10

$f(x) = a_{10}x^{10} + a_9x^9 \ldots\ldots + a_1x + a_0$

$f(-x) = a_{10}x^{10} - a_9x^9 \ldots - a_1x + a_0$

$f(x) + f(-x) = a_{10}x^{10} + a_8x^8 + \ldots + a_0$

Now $g(x) = f'(x) = 10a_{10}x^9 + 9a_9x^8 + \ldots + a_1$

$g(-x) = f'(-x) = -10a_{10}x^9 + 9a_9x^8 + \ldots + a_1$

$g(x) - g(-x) = 20a_{10}x^9 + \ldots$

Clearly degree of $(g(x) - g(-x))$ is 9.

Q.88 A straight line of the form $y = mx + c$ passes through the origin and the point $(x, y) = (2, 6)$. The value of m is______.

[IN, 2016 : 1 Mark]

Solution:

$$y = mx + c$$

passing through (0, 0)

$$0 = 0 + c \Rightarrow c = 0$$

$$y = mx$$

passing through (2, 6)

$\therefore$ $6 = 2\,m$

$\therefore$ $m = 3$

2.7 PARTIAL DERIVATIVES

2.7.1 Definition of Partial Derivative

If a derivative of a function of several independent variables be found with respect to any one of them, keeping the others as constants, it is said to be a partial derivative. The operation of finding the partial derivative of a function of more than one independent variables is called **Partial Differentiation.**

The symbols $\partial/\partial x$, $\partial/\partial y$ etc., are used to denote such differentiations and the expressions $\partial u/\partial x$, $\partial u/\partial y$ etc., are respectively called partial differential coefficients of u with respect to x and y.

If $u = f(x, y, z)$ the partial differential cofficient of u with respect to x i.e., $\partial u/\partial x$ is obtained by differentiating u with respect to x keeping y and z as constants.

2.7.2 Second order partial differential coefficients

If $u = f(x, y)$ then $\partial u/\partial x$ or f_x and $\partial u/\partial y$ or f_y are themselves function of x and y and can be again differentiated partially.

We call $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right), \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)$ as second order partial derivatives of u and these are

respectively denoted by $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}$.

Note: If $u = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial

i.e., $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.89 Let $f = y^x$. What is $\frac{\partial^2 f}{\partial x \partial y}$ at $x = 2, y = 1$?

(a) 0
(b) ln 2
(c) 1
(d) $\frac{1}{\ln 2}$

[ME, GATE-2008, 2 marks]

Solution: (c)

$$f = y^x$$

Treating x as constant, we get

$$\frac{\partial f}{\partial y} = xy^{x-1}$$

Now we treat y as a constant and get,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(y^{x-1}\, x\right) = y^{x-1} + x\, y^{x-1} \ln y$$

whose value at $x = 2$ and $y = 1$ is $= 1^{(2-1)}(1 + 2.\ln 1) = 1$

Q.90 If z = xy ln(xy), then

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

(b) $y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$

(c) $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$

(d) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

[EC, GATE-2014 : 1 Mark, Set-3]

Solution : (c)

$$\frac{\partial z}{\partial x} = y\ln(xy) + \frac{xy}{xy}xy$$

$$\frac{\partial z}{\partial x} = y[\ln(xy) + 1] \quad \text{...(i)}$$

$$\frac{\partial z}{\partial x} = x\ln(xy) + \frac{xy}{xy} \times x$$

$$\frac{\partial z}{\partial x} = x[\ln(xy) + 1] \quad \text{...(ii)}$$

Here $\boxed{x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}}$

Q.91 The contour on the *x*-*y* plane, where the partial derivative of $x^2 + y^2$ with respect to *y* is equal to the partial derivative of $6y + 4x$ with respect to *x*, is

(a) $y = 2$

(b) $x = 2$

(c) $x = y = 4$

(d) $x - y = 0$

[EC, GATE-2015 : 1 Mark, Set-3]

Solution: (a)

Partial derivative w.r.t. *y* $\quad \frac{\partial}{\partial y}(x^2 + y^2) = 2y$

Partial derivative w.r.t. *x* $\quad \frac{\partial}{\partial x}(6y + 4x) = 4$

From given condition $\quad 2y = 4$

$\Rightarrow \quad y = 2$

2.7.3 Homogenous Functions

An expression in which every term is of the same degree is called homogenous function. Thus, $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_{n-1}xy^{n-1} + a_ny^n$ is a homogenous function of x and y of degree n. This can also be written as,

$$x^n\left\{a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + ... + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right\}$$

or $x^n f\left(\frac{y}{x}\right)$, where $f\left(\frac{y}{x}\right)$ is some function of $\frac{y}{x}$.

Note: To test whether a given function f(x, y) is homogenous or not we put tx for x and ty for y in it. If we get $f(tx, ty) = t^n f(x, y)$ the function f(x, y) is homogenous of degree n otherwise f(x, y) is not a homogenous function.

Note: If u is a homogenous function of x and y of degree n then ∂u/∂x and ∂u/∂y are also homogenous function of x and y each being of degree (n − 1).

2.7.4 Euler's Theorem on homogenous functions

If u is a homogenous function of x and y of degree n, then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Note: Euler's theorem can be extended to a homogenous function of any number of variables. Thus if $f(x_1, x_2, \ldots x_n)$ be a homogenous function of $x_1, x_2, \ldots x_n$ of degree n then, $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_n\frac{\partial f}{\partial x_n} = nf$

ILLUSTRATIVE EXAMPLES

Example:

Show that $u = x^3 + y^3 + 3xy^2$ is a homogenous function of degree 3.

Solution:

Now, $$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 \text{ and}$$

$$\frac{\partial u}{\partial x} = 3y^2 + 6xy$$

Now, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = x(3x^2 + 3y^2) + y(3y^2 + 6xy)$$

$$= 3(x^3 + y^3 + 3xy^2)$$

$$= 3u$$

So, Euler's theorem says that u is a homogenous function of degree 3.

2.8 TOTAL DERIVATIVES

If $$u = f(x, y), \text{ where } x = \phi_1(t) \text{ and } y = \phi_2(t),$$

then, $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Here $\frac{du}{dt}$ is called the total differential coefficient of u with respect to t while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u.

In the same way if $u = f(x, y, z)$ where x, y, z are all functions of some variable t, when

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

This result can be extended to any number of variables.

Corollary 1: If u be a function of x and y, where y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

Corollary 2: If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

and $$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}$$

Corollary 3: If x and y are connected by an equation of the form $f(x, y) = 0$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

2.9 MAXIMA AND MINIMA (OF FUNCTION OF TWO INDEPENDENT VARIABLES)

2.9.1 Definitions

Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively.

Then, $f(a, b)$ is said to be maximum and a minimum value of $f(x, y)$ according as $f(a + h, b + k)$ is less or greater than $f(a, b)$ for all sufficiently small independent values of h and k., positive or negative, provided both of them are not equal to zero.

2.9.2 Necessary Conditions

The necessary conditions that $f(x, y)$ should have a maximum or minimum at $x = a$, $y = b$ is that

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}} = 0 \text{ and } \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}} = 0$$

2.9.3 Sufficient Condition for Maxima or Minima

Let $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}}$; $s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a\\y=b}}$; $t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$

Case 1: $f(x, y)$ will have a maximum or a minimum at $x = a$, $y = b$, if $rt > s^2$. Further, $f(x, y)$ is maximum or minimum according as r in negative or positive.

Case 2: $f(x, y)$ will have neither maximum or minimum at $x = a$, $y = b$ if $rt < s^2$, i.e. $x = a$, $y = b$ is a saddle point.

Case 3: If $rt = s^2$ this case is doubtful case and further advanced investigation is needed to determine whether $f(x, y)$ is a maximum or minimum at $x = a$, $y = b$ or not. For gate problems case 3 will not apply. check only case 1 or case 2.

2.10 THEOREMS OF INTEGRAL CALCULUS

1. The integral of the product of a constant and a function is equal to be product of the constant and the integral of function.
   Thus if λ is constant, then $\int \lambda f(x)dx = \lambda \int f(x)dx$.
2. The integral of a sum of or difference of a finite number of functions is equal to sum or difference of integrals. Symbolically
   $$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \dots \pm f_n(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm \dots \pm \int f_n(x)dx$$

2.10.1 Fundamental Formulae

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1}$
2. $\int \frac{1}{x} dx = \log x$
3. $\int \sin x \, dx = \cos x$
4. $\int \cos x \, dx = \sin x$

5. $\int \sec^2 x\, dx = \tan x$
6. $\int \text{cosec}^2 x\, dx = -\cot x$
7. $\int \sec x \tan x = \sec x$
8. $\int \text{cosec}\, x \cot x = -\text{cosec}\, x$
9. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$
10. $\int \frac{1}{1+x^2} dx = \tan^{-1} x$
11. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$
12. $\int \cos hx = \sin hx$
13. $\int \sin hx\, dx = \cos hx$

2.10.2 Useful Trignometric Identities

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	–1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

1. $\sin(-x) = -\sin x$
2. $\cos(-x) = \cos x$
3. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
4. $\sin(x - y) = \sin x \cos y - \cos x \sin y$
5. $\cos(x + y) = \cos x \cos y - \sin x \sin y$
6. $\cos(x - y) = \cos x \cos y + \sin x \sin y$
7. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
8. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
9. (i) $\sin\left(\frac{\pi}{2} + x\right) = \cos x$ (ii) $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
   (iii) $\sin(\pi - x) = \sin x$ (iv) $\cos(\pi - x) = -\cos x$
   (v) $\sin(\pi + x) = -\sin x$ (vi) $\cos(\pi + x) = -\cos x$
   (vii) $\sin(2\pi - x) = -\sin x$ (viii) $\cos(2\pi - x) = \cos x$
10. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
11. $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$
12. $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$
13. $\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$

14. $\cot(x+y) = \dfrac{\cot x \cot y + 1}{\cot y + \cot x}$

15. $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

16. $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

17. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

18. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

19. $\sin^2 x = 1 - \cos^2 x$

20. $\cos^2 x = 1 - \sin^2 x$

21. $e^{it} = \cos t + i \sin t$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.92 Assuming $i = \sqrt{-1}$ and t is a real number, $\int_0^{\pi/3} e^{it} dt$ is

(a) $\frac{\sqrt{3}}{2} + i\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2} - i\frac{1}{2}$

(c) $\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$ (d) $\frac{1}{2} + i\left(1 - \frac{\sqrt{3}}{2}\right)$

[ME, GATE-2006, 2 marks]

Solution: (a)

$$I = \int_0^{\pi/3} e^{it}\, dt = \left[\frac{e^{it}}{i}\right]_0^{\pi/3} = \left[\frac{\cos t + i \sin t}{i}\right]_0^{\pi/3} = \frac{1}{i}\left[\frac{1}{2} + i\frac{\sqrt{3}}{2} - 1\right]$$

$$= \left[-\frac{1}{2i} + \frac{\sqrt{3}}{2}\right] = \left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right]$$

Q.93 The integral $\int_0^{\pi} \sin^3 \theta d\theta$ is given by

(a) 1/2 (b) 2/3

(c) 4/3 (d) 8/3

[EC, GATE-2006, 2 marks]

Solution: (c)

$$I = \int_0^{\pi} \sin^3 \theta . d\theta$$

$\int_0^{\pi} (1-\cos^2\theta) \sin\theta d\theta$, Let $\cos\theta = t$

$-\sin\theta\, d\theta = dt$,

at $\theta = 0$, $t = \cos 0 = 1$

at $\theta = \pi$, $t = \cos\pi = -1$

So, $$I = -\int_{-1}^{+1}(1-t^2)dt = \left|t-\frac{t^3}{3}\right|_{-1}^{1} = \left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)$$

$$I = \frac{2}{3}+\frac{2}{3}=\frac{4}{3}$$

Q.94 If $S = \int_1^{\infty} x^{-3}\,dx$, then S has the value

(a) −1/3
(b) 1/4
(c) 1/2
(d) 1

[EE, GATE-2005, 1 mark]

Solution: (c)

$$\delta = \int_1^{\infty} x^{-3}dx = \left[\frac{x^{-2}}{-2}\right]_1^{\infty} = -\left[\frac{1}{2x^2}\right]_1^{\infty} = -\left[\frac{1}{\infty}-\frac{1}{2}\right] = \frac{1}{2}$$

Q.95 $\int_0^{\pi/4} \frac{(1-\tan x)}{(1+\tan x)}dx$ evaluates to

(a) 0
(b) 1
(c) ln 2
(d) 1/2 ln 2

[CS, GATE-2009, 2 marks]

Solution: (d)

Method 1:

Since, $$\tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}$$

the required integral is $\int_0^{\pi/4}\tan\left(\frac{\pi}{4}-x\right)dx = \ln\sqrt{2} = \frac{1}{2}\ln^2$

Method 2:

Since, $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore \quad I = \int_0^{\pi/4}\frac{1-\tan x}{1+\tan x}dx = \int_0^{\pi/4}\frac{1-\tan\left(\frac{\pi}{4}-x\right)dx}{1+\tan\left(\frac{\pi}{4}-x\right)}$$

Since, $$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\therefore \quad I = \int_0^{\pi/4}\frac{1-\left[\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]}{1+\left[\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]}dx$$

$$= \int_0^{\pi/4} \frac{1-\left[\frac{1-\tan x}{1+\tan x}\right]}{1+\left[\frac{1-\tan x}{1+\tan x}\right]} dx = \int_0^{\pi/4} \frac{(1+\tan x)-(1-\tan x)}{(1+\tan x)+(1-\tan x)} dx$$

$$= \int_0^{\pi/4} \frac{2\tan x}{2} dx = \int_0^{\pi/4} \tan x \, dx$$

$$= [\log(\sec x)]_0^{\pi/4} = \ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec 0)$$

$$= \ln(\sqrt{2}) - \ln(1) = \ln(2^{1/2}) - 0$$

$$= \frac{1}{2}\ln 2$$

Q.96 Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} dx$?

(a) 0
(b) 2
(c) –i
(d) i

Solution: (d) **[CS, GATE-2011, 2 marks]**

$$\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} dx = \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}} dx = \int_0^{\pi/2} e^{2ix} dx$$

$$= \left[\frac{e^{2ix}}{2i}\right]_0^{\pi/2} = \frac{1}{2i}\left[e^{i\pi} - e^{0}\right] = \frac{1}{2i}[-1-1] \left(\text{since } e^{i\pi} = -1\right)$$

$$= \frac{-2}{2i} = \frac{-1}{i} = i$$

2.10.3 Methods of Integration

There are various methods of integration by which we can reduce the given integral to one of the known standard integrals. There are four principal methods of integration.

1. **Integration by substitution:** A change in the variable of integration often reduces an integral to one of fundamental integrals.

Let $I = \int f(x)dx$, then by differentiation w.r.to x we have $\frac{dI}{dx} = f(x)$. Now put,

$$x = \phi(t), \text{ so that } \frac{dx}{dt} = \phi'(t)$$

Then, $$\frac{dI}{dt} = \frac{dI}{dx} \cdot \frac{dx}{dt} = f(x)\cdot\phi'(t) = f\{\phi(t)\cdot\phi'(t)\} \text{ for } x = \phi(t)$$

This gives $$I = \int f\{\phi(t)\cdot\phi'(t)\}dt$$

Rule to Remember:

To evaluate $\int f[\phi(x) \cdot \phi'(x)]dx$

Put $\phi(x) = t$

and $\phi'(x)dx = dt$

where $\phi'(x)$ is the differential coefficient of $\phi(x)$ with respect to x.

Three Forms of Integrals:

(a) $$\int \frac{f'(x)}{f(x)} dx = \log f(x)$$

Put $f(x) = t$ differentiating we get $f'(x) \cdot dx = dt$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log t = \log f(x)$$

Thus the integral of a fraction whose numerator is the exact derivative of its denominator is equal to the logarithmic of its denominator.

Example:

$$\int \frac{4x^3}{1+x^4} dx = \log(1+x^4) \quad \text{... (i)}$$

Because, if we put $(1 + x^4) = t$

$\Rightarrow 4x^3\, dx = dt$

(i) reduces to $\Rightarrow \int \frac{dt}{t} \Rightarrow \log t \Rightarrow \log(1 + x^4)$.

Some Important Formulae Based on the Above Form:

(i) $$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{(-\sin x)}{\cos x} dx$$
$$= -\log \cos x$$
$$= \log(\cos x)^{-1}$$
$$= \log \sec x$$

(ii) $$\int \cot x dx = \log \sin x$$

(iii) $$\int \sec x = \log(\sec x + \tan x)$$

(iv) $$\int \operatorname{cosec} x = \log\left(\tan \frac{x}{2}\right)$$

(b) $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{(n+1)}$ when $n \neq 1$: If the integrand consists of the product of a constant power of a function f(x) and the derivative f′(x) of f(x), to obtain the integral we increase the index by unity and then divide by increased index. This is known as power formula.

Formulae:

(i) $$\int f'(ax+b)dx = \frac{f(ax+b)}{a}$$

(ii) $$\int \frac{1}{\sqrt{a^2+x^2}} dx = \sin h^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2+a^2}\right]$$

(iii) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

(iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2 - a^2}\right]$

(v) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

or $\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left\{x + \sqrt{x^2 + a^2}\right\}$

(vi) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

2. **Integral of the product of two functions**

Integration by parts: Let u and v be two functions of x. Then we have from differential calculus

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \quad \text{...(i)}$$

Integrating both sides of (1) with respect to x, we have

$$uv = \int u \cdot \frac{dv}{dx} dx + \int v \cdot \frac{du}{dx} dx$$

$$\Rightarrow \quad \int u \frac{dv}{dx} dx = uv - \int v \cdot \frac{du}{dx} \cdot dx \quad \text{...(ii)}$$

i.e. $$\int u dv = uv - \int v \cdot du$$

This can also be written as $\int uv dx = u\int v dx - \int [du \int v dx] dx$

The choice of which function will be u and which function will be dv is very important in solving by integration by parts.

The ILATE method helps to decide this.

ILATE stands for

I : Inverse trigonometric functions ($\sin^{-1}x$, $\cos^{-1}x$ etc)

L : Logarithmic functions (log x, ln x etc.)

A : Algebraic functions (x^2, $x^3 + x^2 + 2$, etc.)

T : Trigonometric functions (sinx, cos x etc.)

E : Exponential function (e^x, a^x etc.)

whichever of the two functions comes first in ILATE, get designated as u and other function gets designated as dv.

Formulae Based Upon Above Method:

(a) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$

(b) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$

Integration by Partial Fractions:

(a) $$I = \int \frac{1}{x^2 - a^2}, (x > a)$$

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right\}$$

$$= \frac{1}{2a}\{\log(x-a) - \log(x+a)\} = \frac{1}{2a}\log\frac{x-a}{x+a}$$

Thus $$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\frac{x-a}{x+a}, x > a$$

(b) $$I = \int \frac{1}{a^2 - x^2} dx \ (x < a)$$

In this case $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\frac{a+x}{a-x}, x < a$

The following is a summary of some of the integrals derived so far by using the three methods of integration.

(a) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(b) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\frac{a+x}{a-x}$

(c) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\frac{x-a}{x+a}$

(d) $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sin h^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2 + a^2}\right]$

(e) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$

(f) $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cos h^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2 - a^2}\right]$

(g) $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$

(h) $\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right)$

(i) $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right)$

A few other useful integration formulae:

(a) $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\frac{n+1}{2}}{2\Gamma\left[\frac{m+n+2}{2}\right]}$$

where $\Gamma(x)$ is called the gamma function which satisfies the following properties

$$\Gamma(n+1) = n\,\Gamma n$$

$$\Gamma(n+1) = n! \qquad \text{if n is a positive integer}$$

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(b) Walle's formula

$$\int_0^{\pi/2} \sin^n x = \int_0^{\pi/2} \cos^n x = \begin{cases} \dfrac{(n-1)(n-3)(n-5)}{(n)(n-2)(n-4)} \cdots \dfrac{2}{3}, \text{ when n is odd} \\ \dfrac{(n-1)(n-3)(n-5)}{(n)(n-2)(n-4)} \cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}, \text{ when n is even} \end{cases}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.97 Which of the following integrals is unbounded ?

(a) $\int_0^{\pi/4} \tan x\, dx$

(b) $\int_0^{\infty} \frac{1}{x^2+1} dx$

(c) $\int_0^{\infty} x e^{-x}\, dx$

(d) $\int_0^{1} \frac{1}{1-x} dx$

[ME, GATE-2008, 2 marks]

Solution: (d)

Choice (a) $\int_0^{\frac{\pi}{4}} \tan x\, dx = \log\sqrt{2}$

Choice (b) $\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$

Choice (c) $\int_0^{\infty} x\, e^{-x} dx$

Integrating by parts, taking $u = x$ and $dv = e^{-x}\, dx$ we get $du = dx$ and $v = -e^{-x}$

So, $\int x\, e^{-x}\, dx = x(-e^{-x}) - \int -e^{-x}\, dx = -x\, e^{-x} - e^{-x} = -e^{-x}(x+1)$

Now $\int_0^{\infty} x\, e^{-x} dx = [-e^{-x}(x+1)]_0^{\infty} = 1$

Choice (d) $\int_0^{1} \frac{1}{1-x} dx = \ln 0 - \ln 1 = -\infty - 0 = -\infty$

Since, only (d) is unbounded, (d) is the answer.

Q.98 The value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is

(a) $-\pi$

(b) $-\pi/2$

(c) $\pi/2$

(d) π

[ME, GATE-2010, 1 mark]

Solution: (d)

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_{-\infty}^{\infty}$$

$$= \tan^{-1}(\infty)-\tan^{-1}(-\infty) = \frac{\pi}{2}-\left[\frac{-\pi}{2}\right] = \pi$$

Q.99 The value of the quantity P, where $P=\int_0^1 xe^x dx$, is equal to

(a) 0
(b) 1
(c) e
(d) 1/e

[EE, GATE-2010, 1 mark]

Solution: (b)

$$P = \int_0^1 x\,e^x\,dx\ \ dx$$

Integrating by parts:

Let

$$u = x,$$
$$dv = e^x\,dx$$
$$du = dx,$$
$$v = \int e^x\,dx = e^x$$

Now,

$$\int u\,dv = uv-\int v\,du$$

$\therefore$

$$\int x\,e^x\,dx = x\,e^x-\int e^x\,dx$$
$$= x\,e^x-e^x+c$$

$$\int_0^1 x\,e^x\,dx = [x\,e^x-e^x]_0^1$$
$$= (1\,.\,e^1-e^1)-(0\,.\,e^0-e^0) = 0-(-1)$$
$$= 1$$

Q.100 Let $f(x) = x\,e^{-x}$. The maximum value of the function in the interval $(0, \infty)$ is

(a) e^{-1}
(b) e
(c) $1-e^{-1}$
(d) $1+e^{-1}$

[EE, GATE-2014 : 1 Mark, Set-1]

Solution : (a)

$$f(x) = xe^{-x}$$
$$f'(x) = e^{-x}-x\,e^{-x} = 0$$
$$e^{-x}(1-x) = \infty$$

$\Rightarrow$ $x = 1$ (since $e^{-x} = 0$ only when $x = \infty$ which does not belong to the given interval)

Now, we need to check whether at $x = 1$, we have a maximum, minimum or saddle point.

$$f''(x) = -e^{-x}-(e^{-x}-xe^{-x}) = -2e^{-x}+xe^{-x} = e^{-x}(x-2)$$
$$f''(1) = -e^{-1} \text{ which is } < 0$$

So at $x = 1$, we have a maximum.

The maximum value is $f(1) = 1\,e^{-1} = e^{-1}$

Q.101 Minimum of the real valued function $f(x) = (x-1)^{2/3}$ occurs at x equal to

(a) $-\infty$ (b) 0
(c) 1 (d) ∞

[EE, GATE-2014 : 1 Mark, Set-2]

Solution : (c)

$$f(x) = (x-1)^{2/3} = (\sqrt[3]{x-1})^2$$

As f(x) is square of $\sqrt[3]{x-1}$ hence its minimum value be 0 where at x = 1

Q.102 The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval [-3, 3] is

(a) 20 (b) 28
(c) 16 (d) 32

[EE, GATE-2014 : 2 Marks, Set-2]

Solution : (b)

$$f(x) = x^3 - 3x^2 - 24x + 100 \qquad x \in [-3, 3]$$

$$f'(x) = 3x^2 - 6x - 24$$

$$f'(x) = 0 \quad \text{at } x = 4, -2$$

Critical points are {-3, -2, 3}

$$f(-3) = -27 - 27 + 72 + 100 = 118$$

$$f(-2) = -8 - 12 + 48 + 100 = 128$$

$$f(3) = 27 - 27 - 72 + 100 = 28$$

Hence f(x) has minimum value at x = 3 which is 28.

Q.103 For $0 \le t < \infty$, the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

(a) $t = \log_e 4$ (b) $t = \log_e 2$
(c) $t = 0$ (d) $t = \log_e 8$

[EC, GATE-2014 : 1 Mark, Set-2]

Solution : (a)

$$f(t) = e^{-t} - 2e^{-2t}$$

$$f'(t) = -e^{-t} + 4e^{-2t}$$

For maximum value f′(t) = 0

$$f'(t) = 0 = -e^{-t} + 4e^{-2t}$$

$$\Rightarrow \quad 4e^{-2t} = e^{-t}$$

$$4e^{-t} = 1$$

$$\therefore \quad t = \log_e 4$$

Q.104 The maximum value of the function $f(x) = ln(1 + x) - x$ (where $x > -1$) occur at x = ______.

[EC, GATE-2014 : 1 Mark, Set-3]

Solution :

$$f'(x) = \frac{1}{1+x} - 1 = 0$$

$$\frac{1-1-x}{1+x} = 0$$

$$\frac{x}{1+x} = 0$$

$$x = 0$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f''(0) = -1 < 0$$

f(x) have maximum value at x = 0

$$f(0) = \text{hg}(1 + 0) - 0 = 0$$

$$f_{max} = 0$$

Q.105 The maximum value of

$f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval $0 \le x \le 3$ is

Solution : **[EC, GATE-2014 : 2 Marks, Set-3]**

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$

$$f'(x) = 6x^2 - 18x + 12$$

$$f'(x) = 0$$

$$6x^2 - 18x + 12 = 0$$

$$x^2 = 3x + 2 = 0$$

$$x = 1, 2$$

Hence critical points are {0, 1, 2, 3,}.

f(x) attains its maximum value at one of these points.

$$f(0) = -3$$

$$f(1) = 2$$

$$f(2) = 1$$

$$f(3) = 6$$

Q.106 If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 25$ where $a \ne b$ then $\int_1^2 f(x)dx$ is

(a) $\frac{1}{a^2 - b^2}\left[a(ln2 - 25) + \frac{47b}{2}\right]$

(b) $\frac{1}{a^2 - b^2}\left[a(2ln2 - 25) - \frac{47b}{2}\right]$

(c) $\frac{1}{a^2 - b^2}\left[a(2ln2 - 25) + \frac{47b}{2}\right]$

(d) $\frac{1}{a^2 - b^2}\left[a(ln2 - 25) - \frac{47b}{2}\right]$

[CS, GATE-2015 : 2 Marks, Set-3]

Solution: (a)

$$a\,f(x) + b\,f\left(\frac{1}{x}\right) = \frac{1}{x} - 25 \quad \text{...(1)}$$

Put $x = \frac{1}{x}$ in equation (1)

$$a\,f\left(\frac{1}{x}\right) + b\,f(x) = x - 25 \quad \text{...(2)}$$

Equation (1) × a – equation (2) × b

(1) × a : ⇒ $a^2f(x) + ba\,f\left(\frac{1}{x}\right) = \frac{a}{x} - 25a$

(2) × b : ⇒ $ab\,f\left(\frac{1}{x}\right) + b^2\,f(x) = bx - 25b$

$$a^2f(x) - b^2f(x) = \frac{a}{x} - 25a - bx + 25b$$

$$\Rightarrow \quad (a^2 - b^2) \cdot f(x) = \frac{a}{x} - bx + 25(b-a)$$

$$\Rightarrow \quad f(x) = \frac{1}{a^2 - b^2}\left[\frac{a}{x} - bx + 25(b-a)\right]$$

$$\Rightarrow \quad \int_1^2 f(x)\cdot dx = \frac{1}{a^2 - b^2}\left[a\cdot\int_1^2 \frac{1}{x}\cdot dx - b\int_1^2 x\cdot dx + 25(b-a)\int_1^2 1\cdot dx\right]$$

$$= \frac{1}{a^2 - b^2}\left[a\ln 2 - \frac{3}{2}b + 25(b-a)\right]$$

$$= \frac{1}{a^2 - b^2}\left[a\ln 2 - 25a + \frac{47b}{2}\right]$$

$$= \frac{1}{a^2 - b^2}\left[a(\ln 2 - 25) + \frac{47b}{2}\right]$$

2.11 DEFINITE INTEGRALS

If $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral of f(x) between the limit of a and b.

b → upper limit; a → lower limit.

2.11.1 Fundamental Properties of Definite Integrals

1. We have $\int_a^b f(x)dx = \int_a^b f(t)dt$ i.e., the value of a definite integral does not change with the change of variable of integration provided the limits of integration remain the same.

   Let $\int f(x)dx = F(x)$ and $\int f(t)dt = F(t)$

   Now $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

   $\int_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$

2. $\int_a^b f(x)dt = -\int_b^a f(x)dt$. Interchanging the limits of a definite integral does not change in the absolute value but change the sign of integrals.

3. We have $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

   Note 1: This property also holds true even if the point c is exterior to the interval (a, b).

   Note 2: In place of one additional point c, we can take several points. Thus several points.

   Thus, $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \dots + \int_{c_n}^{b} f(x)dx$

4. (a) We have $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

   (b) We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: Let $$I = \int_0^a f(x)dx$$

Put $x = a - t \Rightarrow dx = -dt$ where $x = 0$, $t = a$ and when $x = a$, $t = 0$

$\Rightarrow$ $$I = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)dt = \int_0^a f(a-x)dx$$

5. $\int_{-a}^{+a} f(x)dx = 0$ or $2\int_0^a f(x)dx$ according as f(x) is an odd or even function of x.

Odd and Even function

(a) An odd function of x if $f(-x) = -f(x)$

(b) An even function of x if $f(-x) = f(x)$.

6. $$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \quad \text{if } f(2a-x) = f(x)$$

and $$\int_0^{2a} f(x)dx = 0, \quad \text{if } f(2a-x) = -f(x)$$

Corollary: $$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

7. $$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$$

if $f(x) = f(x+a)$ [periodic function with period a]

8. $$\frac{d}{dt}\int_{\phi(t)}^{\Psi(t)} f(x)\,dx = f[\Psi(t)]\,\Psi'(t) - f[\phi(t)]\,(p)$$

ILLUSTRATIVE EXAMPLES

Example: 1

Evaluate the following definite integrals:

(a) $\int_{-5}^{5} |x+2|dx$ (b) $\int_{1}^{4} (|x| + |x-3|)dx$.

Solution:

(a) Since for $-5 \le x \le -2$, $x + 2 \le 0$

$\Rightarrow$ $|x+2| = -(x+2)$

and for $-2 \le x \le 5$, $x + 2 \ge 0$

$\Rightarrow$ $|x+2| = x+2$.

$$\therefore \int_{-5}^{5} |x+2|\,dx = \int_{-5}^{-2} |x+2|dx + \int_{-2}^{5} |x+2|dx \quad \text{(Property 3)}$$

$$= \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx = \left[-\frac{x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= (-2+4) - \left(-\frac{25}{2} + 10\right) + \left(\frac{25}{2} + 10\right) - (2-4) = 29.$$

(b) Since for $1 \le x \le 3$, $x \ge 0$, $x - 3 \le 0 \Rightarrow |x| = x$, $|x-3| = -(x-3)$

Also for $3 \le x \le 4$, $x \le 0$, $x - 3 \ge 0 \Rightarrow |x| = x$, $|x-3| = x - 3$.

$$\therefore \quad \int_1^4 (|x|+|x-3|)dx = \int_1^3 (|x|+|x-3|)dx + \int_3^4 (|x|+|x-3|)dx \qquad \text{(Property 3)}$$

$$= \int_1^3 (x-(x-3))dx + \int_2^4 (x+x-3)dx$$

$$= \int_1^3 3dx + \int_3^4 (2x-3)dx$$

$$= 3[x]_1^3 + \left[2.\frac{x^2}{2} - 3x\right]_3^4$$

$$= 3(3-1) + (16-12) - (9-9)$$

$$= 16 + 4 - 0 = 10.$$

Example: 2

Evaluate the following definite integrals:

(a) $\int_{-1}^{2} f(x)dx$ where $f(x) = \begin{cases} 2x+1, & x \le 1 \\ x-5, & x > 1 \end{cases}$ (b) $\int_{-1}^{1} \frac{|x|}{x}dx$ (c) $\int_0^1 [3x]dx$

Solution:

(a) First note that the given function is discontinuous at $x = 1$.

$$\therefore \quad \int_{-1}^{2} f(x)dx = \int_{-1}^{1} f(x)dx + \int_1^2 f(x)dx \qquad \text{(Property 3)}$$

$$= \int_{-1}^{1} (2x+1)dx + \int_1^2 (x-5)dx$$

$$= \left[x^2+x\right]_{-1}^{1} + \left[\frac{x^2}{2} - 5x\right]_1^{-2}$$

$$= (1+1)-(1-1)+(2-10)-\left(\frac{1}{2}-5\right) = 2-0-8+\frac{9}{2} = -\frac{3}{2}$$

(b) First note that $\frac{|x|}{x}$ is discontinuous at $x = 0$.

$$\therefore \quad \int_{-1}^{1} \frac{|x|}{x}dx = \int_{-1}^{0} \frac{|x|}{x}dx + \int_0^1 \frac{|x|}{x}dx = \int_{-1}^{0} \frac{-x}{x}dx + \int_0^1 \frac{x}{x}dx$$

$$(\because -1 \le x \le 0 \Rightarrow |x| = -x \text{ and } 0 \le x \le 1 \Rightarrow |x| = x)$$

$$= \int_{-1}^{0} -1dx + \int_0^1 1dx = [-x]_{-1}^{0} + [x]_0^1$$

$$= -(0-(-1)) + (1-0) = -1 + 1 = 0.$$

(c) First note that $[3x]$ is discontinuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$.

$$\therefore \quad \int_0^1 [3x]\,dx = \int_0^{1/3} [3x]\,dx + \int_{1/3}^{2/3} [3x]dx + \int_{2/3}^{1} [3x]dx$$

$$= \int_0^{1/3} 0\,dx + \int_{1/3}^{2/3} 1\,dx + \int_{2/3}^{1} 2\,dx = 0 + [x]_{1/3}^{2/3} + 2[x]_{2/3}^{1}$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) + 2\left(1 - \frac{2}{3}\right) = \frac{1}{3} + \frac{2}{3} = 1$$

Example: 3

By using properties of definite integral, evaluate the following:

(a) $\int_{-\pi/2}^{\pi/2} \sin^4 x\,dx$ (b) $\int_{-\pi/4}^{\pi/2} x^3 \sin^4 x\,dx$ (c) $\int_0^{2\pi} |\cos x|\,dx$

Solution:

(a) Let $f(x) = \sin^4 x \Rightarrow f(-x) = \sin^4(-x) = (-\sin x)^4 = \sin^4 x = f(x)$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \sin^4 x\,dx = 2\int_0^{\pi/2} \sin^4 x\,dx = 2\int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (3 - 4\cos 2x + \cos 4x)\,dx$$

$$= \frac{1}{4}\left[3x - 4.\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(3\frac{\pi}{2} - 2\sin\pi + \frac{1}{4}\sin 2\pi\right) - \left(0 - 2\sin 0 + \frac{1}{4}\sin 0\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{3\pi}{2} - 0 + 0\right) - (0 - 0 + 0)\right] = \frac{3\pi}{8}$$

(b) Let $f(x) = x^3 \sin^4 x \Rightarrow f(-x) = (-x)^3 \sin^4(-x) = -x^3 \sin^4 x = -f(x)$

$\Rightarrow f(x)$ is an odd function; therefore, by property 5,

$$\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx = 0$$

(c) Let $f(x) = |\cos x| \Rightarrow f(2\pi - x) = |\cos(2\pi - x)| = |\cos x| = f(x)$, therefore, by property 6,

$$\int_0^{2\pi} |\cos x|\,dx = 2\int_0^{\pi} |\cos x|\,dx \quad \text{... (i)}$$

Again, $f(\pi - x) = |\cos(\pi - x)| = |-\cos x| = |\cos x| = f(x)$, therefore, by property 6,

$$\int_0^{\pi} |\cos x|\,dx = 2\int_0^{\pi/2} |\cos x|\,dx \quad \text{... (ii)}$$

∴ From (i) and (ii), we get

$$\int_0^{2\pi} |\cos x| dx = 2.2\int_0^{2\pi} |\cos x| dx = 4\int_0^{\pi/2} \cos x\, dx$$

$$\left(\because \text{ for } 0 \le x \le \frac{\pi}{2}, \cos x \ge 0 \Rightarrow |\cos x| = \cos x\right)$$

$$= 4[\sin x]_0^{\pi/2} = 4\left(\sin\frac{\pi}{2} - \sin 0\right) = 4(1-0) = 4.$$

Example: 4

Evaluate the following $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$.

Solution:

Let $$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \quad \text{...(i)}$$

Then, by using property 4b, we get

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \text{...(ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

Example: 5

Evaluate the following definite integrals:

(a) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

(b) $\int_0^{\pi/2} \sin 2x \log(\tan x) dx$

Solution:

(a) Let $$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \text{...(i)}$$

Then, by using property 4b, we get

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx = \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right)^{-1} dx = \int_0^1 -1.\log\left(\frac{1-x}{x}\right) dx = -\int_0^1 \log\left(\frac{1-x}{x}\right) dx$$

$$= -I$$

$$\Rightarrow \quad 2I = 0$$

$$\Rightarrow \quad I = 0$$

(b) Let $$I = \int_0^{\pi/2} \sin 2x \log(\tan x) dx \quad \dots (i)$$

Then, by using property 4b, we get

Let $$I = \int_0^{\pi/2} \sin\left(2\left(\frac{\pi}{2} - x\right)\right)\log\left(\tan\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x)\log(\cot x)dx = \int_0^{\pi/2} \sin 2x \log\left((\tan x)^{-1}\right)dx$$

$$= \int_0^{\pi/2} \sin 2x(-1)\log(\tan x)dx = \int_0^{\pi/2} \sin 2x \log(\tan x)dx$$

$$= -I \quad \text{[using (i)]}$$

$$\Rightarrow \quad 2I = 0$$

$$\Rightarrow \quad I = 0$$

Example: 6

Evaluate the following definite integrals $\int_0^{\pi} \log(1 + \cos x)\, dx$.

Solution:

$$I = \int_0^{\pi} \log(1 + \cos x)dx \quad \dots (i)$$

Then, by using property 4b, we get

$$I = \int_0^{\pi} \log(1 + \cos(\pi - x))dx = \int_0^{\pi} \log(1 - \cos x)dx \quad \dots (ii)$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi} (\log(1 + \cos x) + \log(1 - \cos x))dx = \int_0^{\pi} \log\left(1 - \cos^2 x\right)dx$$

$$= \int_0^{\pi} \log\left(\sin^2 x\right)dx = 2\int_0^{\pi} \log \sin x \, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \log \sin x \, dx$$

Let $f(x) = \log \sin x \Rightarrow f(\pi - x) = \log(\sin(\pi - x)) = \log \sin x = f(x)$, therefore, by using property 6, we get

$$I = 2\int_0^{\pi/2} \log \sin x \, dx = 2.\left(-\frac{\pi}{2}\log 2\right) = -\pi \log 2.$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.107 What is the value of the definite integral, $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$?

(a) 0 (b) a/2

(c) a (d) 2a [CE, GATE-2011, 2 marks]

Solution: (b)

Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ ...(i)

Since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx$ ...(ii)

(i) + (ii) ⇒ $2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$

⇒ $2I = \int_0^a dx$

⇒ $2I = a$

⇒ $I = a/2$

Q.108 The value of $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$ is

(a) 0 (b) $\frac{1}{15}$

(c) 1 (d) $\frac{8}{3}$

[CE, GATE-2013, 2 Mark]

Solution: (b)

Let

$3\theta = t$

$3 \times d\theta = dt$

$\therefore d\theta = \frac{dt}{3}$

$\theta = \frac{\pi}{6} \quad t = \frac{\pi}{2}$

$\theta = 0 \quad t = 0$

$$I = \int_0^{\pi/2} \cos^4 t \cdot \sin^3 2t \cdot \frac{dt}{3} = \frac{1}{3}\int_0^{\pi/2} \cos^4 t \cdot (2\sin t \cos t)^3 \cdot dt$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^4 t \cdot \sin^3 t \cdot \cos^3 t dt = \frac{8}{3}\int_0^{\pi/2} \cos^7 t \sin^3 t dt$$

$$= \frac{8}{3}\left[\frac{6\cdot4\cdot2\cdot2}{10\cdot8\cdot6\cdot4\cdot2}\right] = \frac{1}{15}$$

Q.109 If f(x) is an even function and a is a positive real number, then $\int_{-a}^{a} f(x)dx$ equals

(a) 0
(b) a
(c) 2a
(d) $2\int_0^a f(x)dx$

[ME, GATE-2011, 1 marks]

Solution: (d)

If f(x) is even function then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

Q.110 The value of the definite integral $\int_1^e \sqrt{x}\ln(x)dx$ is

(a) $\frac{4}{9}\sqrt{e^3}+\frac{2}{9}$
(b) $\frac{2}{9}\sqrt{e^3}-\frac{4}{9}$
(c) $\frac{2}{9}\sqrt{e^3}+\frac{4}{9}$
(d) $\frac{4}{9}\sqrt{e^3}-\frac{2}{9}$

[ME, GATE-2013, 2 Marks]

Solution: (c)

$$I = \int_1^e \sqrt{x}\, ln x\, dx$$

$$u = ln x \quad ; \quad dv = \sqrt{x}dx$$

$$du = \frac{1}{x}dx \quad ; \quad dv = \int \sqrt{x}\, dx = \frac{x^{3/2}}{\frac{3}{2}}$$

$$\int udv = uv - \int vdu$$

$$\int_1^e ln x \cdot \sqrt{x}\, dx = \left(ln x \cdot \frac{x^{3/2}}{3/2} \right)\Bigg|_1^e - \int_1^e \frac{x^{3/2}}{3/2} \cdot \frac{1}{x} dx$$

$$= \left(\frac{2}{3}e^{3/2} - 0\right) - \frac{2}{3}\int_1^e x^{1/2}dx = \frac{2}{3}e^{3/2} - \frac{2}{3}\left(\frac{x^{3/2}}{3/2}\right)\Bigg|_1^e$$

$$= \frac{2}{3}e^{3/2} - \frac{4}{9}(e^{3/2}-1) = \frac{2}{9}e^{3/2} + \frac{4}{9}$$

Q.111 The value of the inegral $\int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^2 + \cos(x-1)} dx$ is

(a) 3
(b) 0
(c) −1
(d) −2

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (b)

$$I = \int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^2 + \cos(x-1)} dx$$

Taking $x - 1 = z \Rightarrow dx = dz$

for $x = 0$, $z \to -1$ and $x = 2$, $z \to 1$

$$\therefore \qquad I = \int_{-1}^{1} \frac{z^2 \sin z}{z^2 + \cos z} dz$$

let
$$f(z) = \frac{z^2 \sin z}{z^2 + \cos z}$$

$$f(-z) = \frac{z^2 \sin z}{z^2 + \cos z}$$

$$f(z) = -f(z) \text{ function is ODD.}$$

$$\therefore \qquad I = 0$$

Q.112 If $\int_0^{2\pi} |x \sin x| dx = k\pi$, then the value of k is equal to ______.

[CS, GATE-2014 : 1 Mark, Set-3]

Solution :

$$\Rightarrow \int_0^{2\pi} |x \sin x| dx = Kp$$

$$\Rightarrow \int_0^{\pi} |x \sin x| dx + \int_{\pi}^{2\pi} |x \sin x| dx = Kp$$

$$\Rightarrow \int_0^{\pi} x \sin x\, dx + \int_{\pi}^{2\pi} -(x \sin x) dx = Kp$$

$$\Rightarrow (-x \sin x + \sin x)|_0^{\pi} - (-x \sin x + \sin x)|_{\pi}^{2\pi} = Kp$$

$$\Rightarrow 4p = Kp$$

$$\Rightarrow K = 4$$

Q.113 The value of the integral given below is

$$\int_0^{\pi} x^2 \cos x\, dx$$

(a) -2π (b) π

(c) $-\pi$ (d) 2π

[CS, GATE-2014 : 2 Marks, Set-3]

Solution : (a)

$$\int_0^{\pi} x^2 \cos x\, dx = x^2 (\sin x) - 2x(-\cos x) + 2(-\sin x)|_0^{\pi} = \pi^2 \cdot 0 + 2\pi(-1) - 0 = -2\pi$$

Q.114 The integral $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$ is equal to ______.

[EC, 2016 : 1 Mark, Set-3]

Solution:

$$\int_0^1 \frac{1}{\sqrt{1-x}} dx = -2\int_0^1 \frac{1}{2\sqrt{1-x}} dx = -2\left(\sqrt{1-x}\right)|_0^1 = -2(0-1) = 2$$

Q.115 The value of $\int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx$ is

(a) $\frac{\pi}{2}$
(b) π
(c) $\frac{3\pi}{2}$
(d) 1

[CE, 2016 : 2 Marks, Set-I]

Solution: (b)

$$\int_0^\infty \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^\infty = \tan^{-1}\infty - \tan^{-1}0 = \frac{\pi}{2}$$

and $$L(\sin x) = \frac{1}{s^2+1}$$

$\Rightarrow$ $$L\left(\frac{\sin x}{x}\right) = \int_s^\infty \frac{1}{s^2+1}dx \quad \text{(Using "division by } x\text{")}$$

$$= \left[\tan^{-1}s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}(s) = \cot^{-1}(s)$$

$\Rightarrow$ $$\int_0^\infty e^{-sx}\frac{\sin x}{x}dx = \cot^{-1}(s) \quad \text{(Using definition of Laplace transform)}$$

Put $s = 0$, we get $$\int_0^\infty \frac{\sin x}{x}dx = \cot^{-1}(0) = \frac{\pi}{2}$$

$$\frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx = \pi$$

2.12 APPLICATIONS OF INTEGRATION

We study three areas where integration is applied

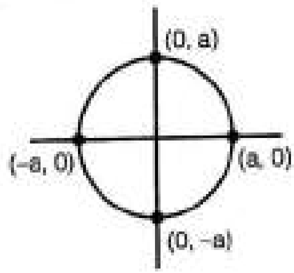
1. Areas of curves
2. Length of curves
3. Volumes of revolution

2.12.1 Preliminary : Curve Tracing

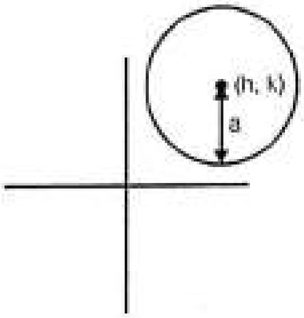
In order to find area under curves, as well as for evaluating double and triple integrals, it is used to know how to trace some common curves from their equations.

Circle : Cartesian Form:

1. $x^2 + y^2 = a^2$: Circle with centre (0, 0) and radius a.



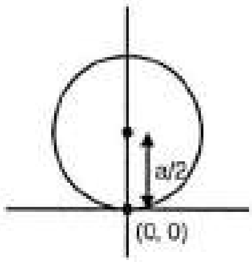
2. $(x-h)^2 + (y-k)^2 = a^2$: Circle with centre (h, k) and radius a.



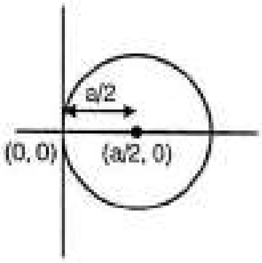
Polar Form:

1. $r = a$: Circle with centre (0, 0) and radius a.

2. $r = a \sin\theta$: Circle with centre $\left(0, \frac{a}{2}\right)$ and radius $\frac{a}{2}$.

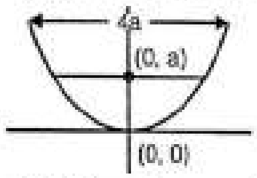


3. $r = a \cos\theta$: Circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

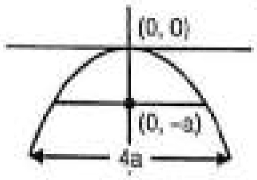


Parabola:

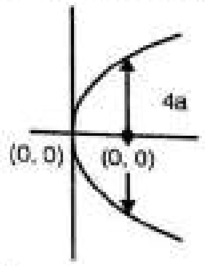
1. $x^2 = 4ay$: Parabola with vertex at (0, 0) and focus at (0, a) and latus rectum = 4a.



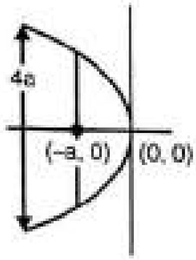
2. $x^2 = -4ay$: Parabola with vertex at (0, 0) and focus at (0, −a) and latus rectim = 4a.



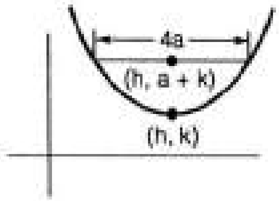
3. $y^2 = 4ax$: Parabola with vertex at (0, 0) and focus at (a, 0) and latus rectum = 4a.



4. $y^2 = -4ax$: Parabola with vertex at (0, 0) and focus at (–a, 0) and latus rectum = 4a.

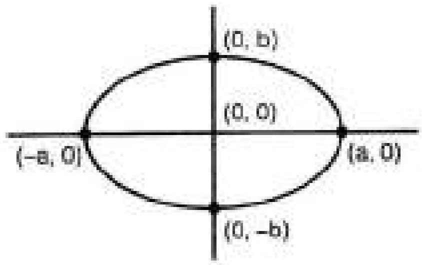


5. $(x - h)^2 = 4a(y - k)$: Parabola with centre at (h, k) focus at (0 + h, a + k) and latus rectum = 4a.

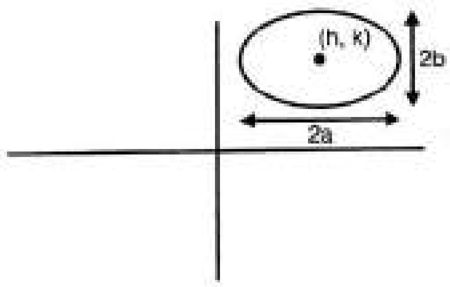


Ellipse:

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$: Ellipse with centre at (0, 0) and major axis = 2a and minor axis = 2b.

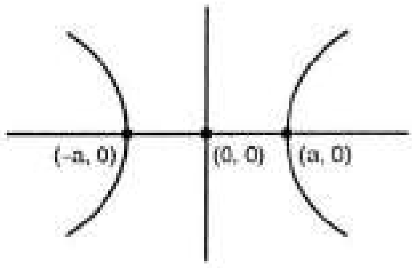


2. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$: Ellipse with centre at (h, k) and major axis = 2a and minor axis = 2b.

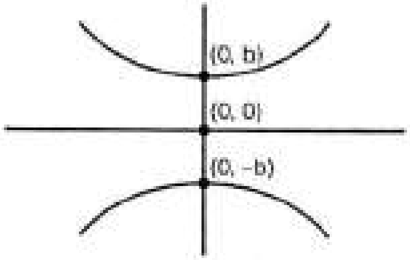


Hyperbola:

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$: Hyperbola with vertex at (a, 0) and (–a, 0) and centre at (0, 0).



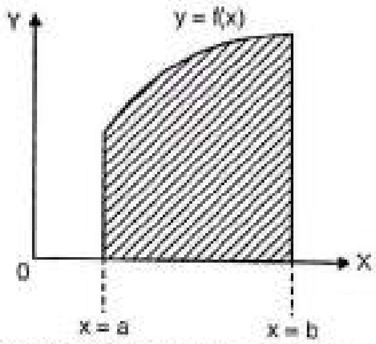
2. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$: Hyperbola with vertex at (0, b) and (0, –b) and centre at (0, 0).



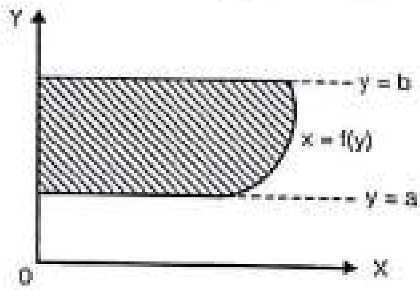
2.12.2 Areas of Cartesian Curves

Theorem: Area bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b is

$$\int_a^b y\,dx = \int_a^b f(x)\,dx$$

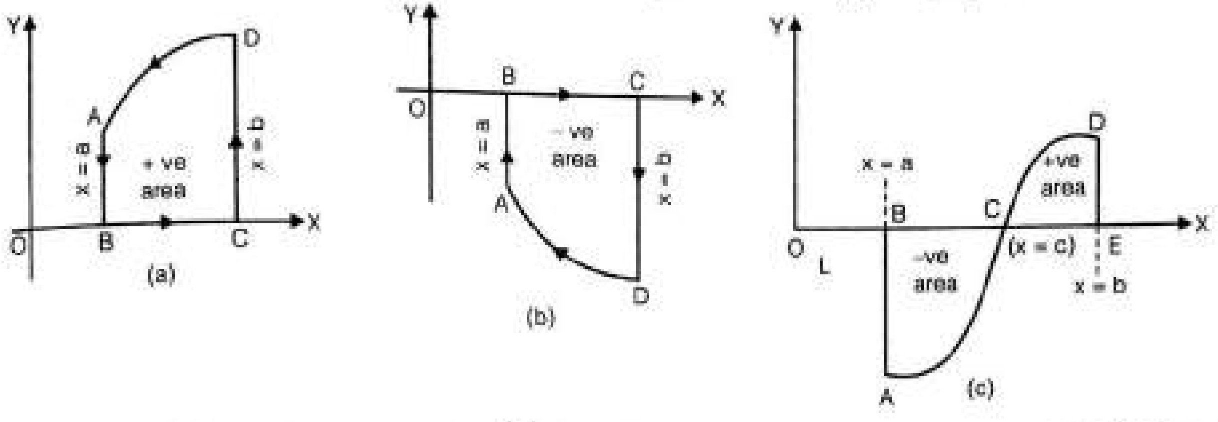


2. Interchanging x and y in the above formula, we see that the area bounded by the curve x = f(y), the x-axis and the abscissae y = a, y = b is $\int_a^b x dx = \int_a^b f(y) dy$ as shown in figure below.



Note. 1 : The area bounded by a curve, the x-axis and two ordinates is called the **area under the curve.** The process of finding the area of plane curves is often called **quadrature.**

Note. 2 : Sign of an area. an area whose boundary is described in the anti-clockwise direction (i.e. lies above x-axis) is considered positive (Fig. a) and an area whose boundary is described in the clockwise direction (i.e. lies below x-axis) is taken as negative (Fig. b).



In Fig. (c) above, the area given by $\int_a^b y\,dx$ will not consist of the sum of the area ABC $\left(=\int_a^c y\,dx\right)$ and the area CDE $\left(=\int_c^b y\,dx\right)$ but their difference.

Thus to find the total area in such cases the numerical value of the area of each portion must be evaluated separately by taking modulus and their results added afterwards.

ILLUSTRATIVE EXAMPLES

Example:

Find the area of the segment cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$.

Solution:

Given parabola is $\quad x^2 = 8y \quad$ ... (i)

and the straight line is

$$x - 2y + 8 = 0$$

$$\Rightarrow \quad y = \frac{x+8}{2} \quad \text{... (ii)}$$

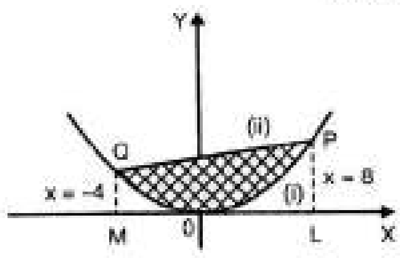
Substituting the value of y from (ii) in (i), we get

$$x^2 = 4(x+8)$$

or $\quad x^2 - 4x - 32 = 0$

or $\quad (x-8)(x+4) = 0$

$\therefore \quad x = 8, -4$



Thus (i) and (ii) intersect at P and Q where x = 8 and x = – 4.

$\therefore$ Required area POQ (i.e. dotted area) = [area bounded by st. line (ii) and x-axis from x = – 4 to x = 8] – [area bounded by parabola (i) and x-axis from x = – 4 to x = 8]

$$= \int_{-4}^{8} y\,dx, \text{ from (ii)} - \int_{-4}^{8} y\,dx, \text{ from (i)}$$

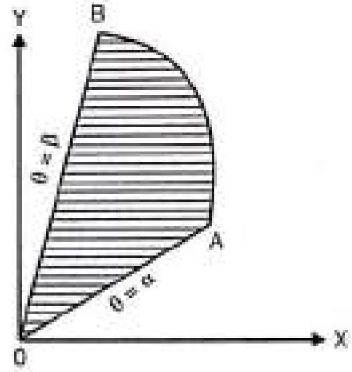
$$= \int_{-4}^{8} \frac{x+8}{2}dx - \int_{-4}^{8} \frac{x^2}{8}dx = \frac{1}{2}\left|\frac{x^2}{2}+8x\right|_{-4}^{8} - \frac{1}{8}\left|\frac{x^3}{3}\right|_{-4}^{8}$$

$$= \frac{1}{2}\{(32+64)-(-24)\} - \frac{1}{24}(512+64) = 36.$$

2.12.3 Areas of Polar Curves

Theorem: Area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$



ILLUSTRATIVE EXAMPLES

Example:

Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.

Solution:

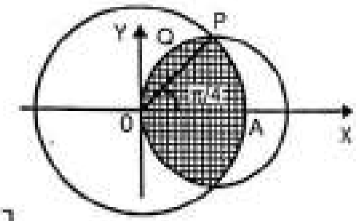
The equations of the circles are

$$r = a\sqrt{2} \text{ and} \quad \text{... (i)}$$
$$r = 2a\cos\theta \quad \text{... (ii)}$$

(i) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$.

(ii) represents a circle symmetrical about OX, with centre at (a, 0) and radius a.

The circles are shown in Fig. below. At their point of intersection P, eliminating r from (i) and (ii),



$$a\sqrt{2} = 2a\cos\theta \text{ i.e., } \cos\theta = \frac{1}{\sqrt{2}}$$

or $\theta = \pi/4$

$\therefore$ Required area $= 2 \times$ area OAPQ (by symmetry)

$= 2$(area OAP + area OPQ)

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2 d\theta, \text{ for (i)} + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2 d\theta, \text{ for (ii)}\right]$$

$$= \int_0^{\pi/4}\left(a\sqrt{2}\right)^2 d\theta + \int_{\pi/4}^{\pi/2}(2a\cos\theta)^2 d\theta$$

$$= 2a^2 |\theta|_0^{\pi/4} + 4a^2\int_{\pi/4}^{\pi/2}\frac{1+\cos 2\theta}{2} d\theta$$

$$= 2a^2(\pi/4 - 0) + 2a^2\left|\theta + \frac{\sin 2\theta}{2}\right|_{\pi/4}^{\pi/2}$$

$$= \frac{\pi a^2}{2} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = a^2(\pi - 1).$$

2.12.4 Derivative of arc Length δ

Theorem: For the curve y = f(x), we have

$$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$

Proof: Let P(x, y), Q(x + δx, y + δy) be two neighbouring points on the curve AB (Figure below). Let arc AP = s, arc PQ = δs.

Draw PL, QM ⊥s on the x-axis and PN ⊥ QM.

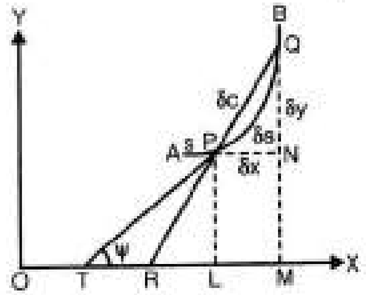
∴ From the rt. triangle PNQ,

$$PQ^2 = PN^2 + NQ^2$$

i.e. $$\delta c^2 = \delta x^2 + \delta y^2$$

or $$\left(\frac{\delta c}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

∴ $$\left(\frac{\delta s}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\right)^2\left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]$$



Taking limits as Q → P (i.e. δc → 0),

$$\left(\frac{ds}{dx}\right)^2 = 1.\left[1+\left(\frac{dy}{dx}\right)^2\right] \qquad \left[\text{Since, } \underset{x\to 0}{Lt}\frac{\delta s}{\delta c} = 1\right]$$

If s increases with x as in Figure above, dy/dx is positive.

Thus $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$, taking positive sign before the radical ... (i)

Cor. 1. If the equation of the curve is x = f(y), then

∴ $$\frac{ds}{dy} = \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]} \quad \text{... (ii)}$$

Cor. 2. If the equation of the curve is in parametric form x = f(t), y = φ(t), then

$$\frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{dt} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2\right]}$$

∴ $$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]} \quad \text{... (iii)}$$

2.12.5 Lengths of Curves

Theorem: The length of the arc of the curve y = f (x) between the points where x = a and x = b is

$$s = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

The length of the arc of the curve x = f(y) between the points where y = a and y = b, is

$$\int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}dy$$

The length of the arc of the curve x = f(t), y = f(t) between the points where t = a and t = b, is

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}dt$$

The length of the arc of the curve r = f(θ), between the points where θ = α and θ = β, is

$$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}d\theta$$

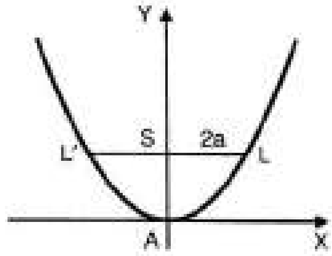
ILLUSTRATIVE EXAMPLES

Example:

Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.

Solution:

Let A be the vertex and L an extremity of the latus-rectum so that at A, x = 0 and at L, x = 2a, as shown in figure.



Now, $y = x^2/4a$

so that $\frac{dy}{dx} = \frac{1}{4a}.2x = \frac{x}{2a}$

$$\therefore \quad \text{arc } AL = \int_0^{2a}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx = \int_0^{2a}\sqrt{\left[1+\left(\frac{x}{2a}\right)^2\right]}dx$$

$$= \frac{1}{2a}\int_0^{2a}\sqrt{[(2a)^2+x^2]}dx = \frac{1}{2a}\left[\frac{x\sqrt{[(2a)^2+x^2]}}{2}+\frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a}$$

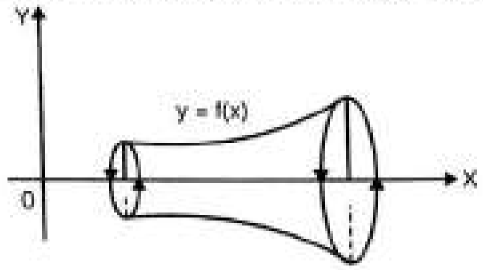
$$= \frac{1}{2a}\left[\frac{2a\sqrt{(8a)^2}}{2}+2a^2\sinh^{-1}1\right]$$

$$= a\left[\sqrt{2}+\sinh^{-1}1\right] = a\left[\sqrt{2}+\log(1+\sqrt{2})\right] \qquad \left[\because \sinh^{-1}x = \log[x+\sqrt{(1+x^2)}\right]$$

2.12.6 Volumes of Revolution

1. **Revolution about x-axis:** The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b \pi y^2 dx$.

Let AB to the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$.

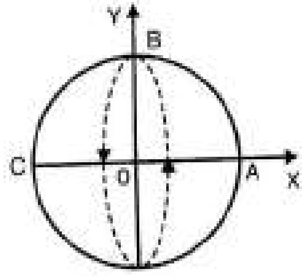


ILLUSTRATIVE EXAMPLES

Example:

Find the volume of a sphere of radius a.

Solution:



Let the sphere be generated by the revolution of the semi-circle ABC, of radius a about its diameter CA, (Figure)

Taking CA as the x-axis and its mid-point O as the origin, the equation of the circle ABC is

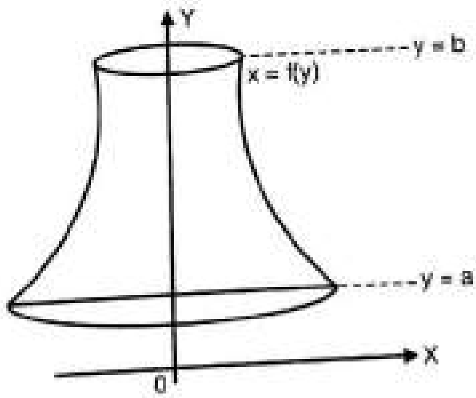
$$x^2 + y^2 = a^2$$

$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

$$= 2\int_0^a \pi y^2 dx = 2\pi\int_0^b (a^2 - x^2)dx$$

$$= 2\pi\left|a^2x - \frac{x^3}{3}\right|_0^a = 2\pi\left|a^3 - \frac{a^3}{3} - (0-0)\right| = \frac{4}{3}\pi a^3$$

2. **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the volume of the solid generated by the revolution, about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is $\int_a^b \pi x^2 dy$.



ILLUSTRATIVE EXAMPLES

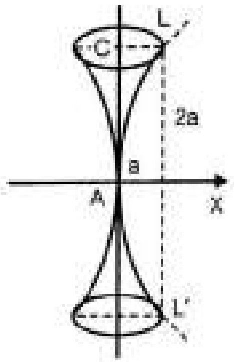
Example:

Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum.

Solution:

Given parabola is $x = y^2/4a$.

Let A be the vertex and L one extremity of the latus-rectum. For the arc AL, y varies from 0 to 2a (Figure)



$\therefore$ Required volume = 2 (volume generated by the revolution about the y-axis of the area ALC)

$$= 2\int_0^{2a} \pi x^2 dy = 2\pi\int_0^{2a} \frac{y^4}{16a^2} dy = \frac{\pi}{8a^2}\left|\frac{y^5}{5}\right|_0^{2a}$$

$$= \frac{\pi}{40a^2}(32a^5 - 0) = \frac{4\pi a^3}{5}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.116 What is the area common to the circles $r = a$ and $r = 2a\cos\theta$?

(a) $0.524\,a^2$
(b) $0.614\,a^2$
(c) $1.047\,a^2$
(d) $1.228\,a^2$

[CE, GATE-2006, 2 marks]

Solution: (d)

Area common to circles, $r = a$
and $r = 2a\cos\theta$ is $1.228a^2$.

Q.117 A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h(x^2/L^2)$, where x is the horizontal coordinate and y is the vertical coordinate with the origin at the centre of the cable. The expression for the total length of the cable is

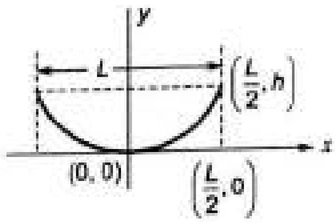
(a) $\int_0^L \sqrt{1+64\frac{h^2x^2}{L^4}}dx$
(b) $2\int_0^{L/2} \sqrt{1+64\frac{h^3x^2}{L^4}}dx$
(c) $\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}dx$
(d) $2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}dx$

[CE, GATE-2010, 2 marks]

Solution: (d)

Length of curve y = f(x) between x = a and x = b is given by

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



here, $y = 4h\frac{x^2}{L^2}$ ... (i)

$$\frac{dy}{dx} = 8h\frac{x}{L^2}$$

since, $y = 0$ at $x = 0$

and $y = h$ at $x = \frac{L}{2}$

(As can be seen from equation (i), by substituting x = 0 and x = L/2)

$$\therefore \quad \frac{1}{2}(\text{Length of cable}) = \int_0^{L/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{L/2} \sqrt{1+\left(\frac{8hx}{L^2}\right)^2}\,dx$$

$$\text{Length of cable} = 2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$$

Q.118 The length of the curve $y = \frac{2}{3}x^{3/2}$ between x = 0 and x = 1 is

(a) 0.27
(b) 0.67
(c) 1
(d) 1.22

[ME, GATE-2008, 2 marks]

Solution: (d)

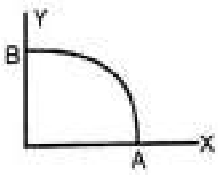
$$y = \frac{2}{3}x^{3/2}$$

$$\frac{dy}{dx} = x^{1/2}$$

length of the curve is given by

$$\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_{x=0}^{x=1} = 1.22$$

Q.119 A path AB in the form of one quarter of a circle of unit radius is shown in the figure. Integration of $(x + y)^2$ on path AB traversed in a counter-clockwise sense is



(a) $\frac{\pi}{2} - 1$
(b) $\frac{\pi}{2} + 1$
(c) $\frac{\pi}{2}$
(d) 1

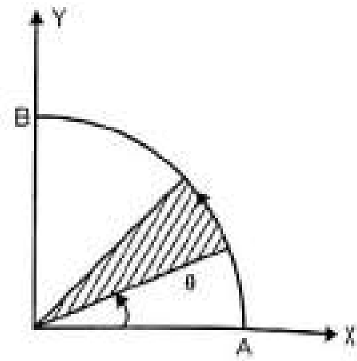
[ME, GATE-2009, 2 marks]

Solution: (b)

Path AB:
$$x^2 + y^2 = 1$$
$$x = \cos\theta$$
$$y = \sin\theta$$

Along path AB θ varies from 0° to 90° [0 to π/2]

$$\int_{\text{Path AB}} (x+y)^2 (r\,d\theta) = \int_0^{\pi/2} (\cos\theta + \sin\theta)^2\, 1.d\theta$$
$$= \int_0^{\pi/2} (\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta)d\theta$$
$$= \int_0^{\pi/2} (1+\sin 2\theta)d\theta$$
$$= \theta + \frac{(-\cos 2\theta)}{2}\Big|_0^{\pi/2}$$
$$= \frac{\pi}{2} - \frac{1}{2}\left[\cos 2\frac{\pi}{2} - \cos 0\right]$$
$$= \frac{\pi}{2} - \frac{1}{2}[-1-1] = \frac{\pi}{2} + 1$$



Q.120 The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is

(a) $\frac{16}{3}$ (b) 8

(c) $\frac{32}{3}$ (d) 16

[ME, GATE-2009, 2 marks]

Solution: (a)

Curve 1 : $y^2 = 4x$

Curve 2 : $x^2 = 4y$

Intersection points of curve 1 and 2

$$y^2 = 4x = 4\sqrt{4y} = 8\sqrt{y}$$
$$y^4 = 8 \times 8\,y \Rightarrow y(y^3 - 64) = 0$$

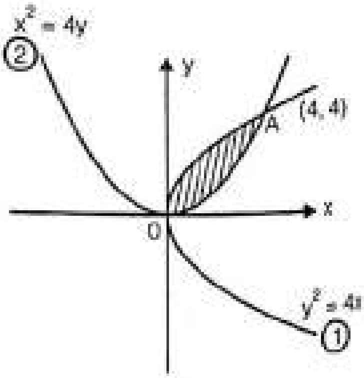
Solution $y = 4$ and $y = 0$

then $x = 4$ $x = 0$

Therefore intersection point are A(4, 4) and O(0, 0)

The area enclosed between curves 1 and 2 are given by

$$\text{Area} = \int_{x_1}^{x_2} y_1\,dx - \int_{x_1}^{x_2} y_2\,dx$$
$$= -\int_0^4 \frac{x^2}{4}dx$$
$$= 2\frac{x^{3/2}}{3/2}\Big|_0^4 - \frac{x^3}{3\times 4}\Big|_0^4 = \frac{4}{3}(4)^{3/2} - \frac{(4)^3}{3\times 4} = \frac{16}{3}$$



Alternately, the same answer could have been obtained by taking a double integral as follows:

$$\text{Required Area} = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\,dy = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \frac{16}{3}$$

Q.121 The parabolic arc $y = \sqrt{x}$, $1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is

(a) $\pi/4$
(b) $\pi/2$
(c) $3\pi/4$
(d) $3\pi/2$

Solution: (d) **[ME, GATE-2010, 1 mark]**

The volume of a solid generated by revolution about the x-axis, of the area bounded by curve $y = f(x)$, the x-axis and the ordinates $x = a$, $y = b$ is

$$\text{Volume} = \int_a^b \pi y^2 dx$$

Here, $a = 1$, $b = 2$ and $y = \sqrt{x} \Rightarrow y^2 = x$

$$\therefore \quad \text{Volume} = \int_1^2 \pi \cdot x \cdot dx$$

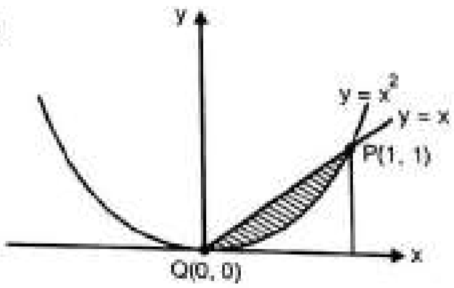
$$= \pi \cdot \left[\frac{x^2}{2}\right]_1^2 = \frac{\pi}{2}\left[x^2\right]_1^2 = \frac{\pi}{2}\left[2^2 - 1^2\right] = \frac{3}{2}\pi$$

Q.122 The area enclosed between the straight line $y = x$ and the parabola $y = x^2$ in the x-y plane is

(a) 1/6
(b) 1/4
(c) 1/3
(d) 1/2

Solution: (a) **[ME, GATE-2012, 1 mark]**

The area enclosed is shown below as shaded:



The coordinates of point P and Q is obtained by solving

$$y = x$$

and $\quad y = x^2$ simultaneously,

i.e. $\quad x = x^2$

$\Rightarrow \quad x(x - 1) = 0$

$\Rightarrow \quad x = 0, \quad x = 1$

Now, $x = 0 \Rightarrow y = 0$ which is pt Q(0, 0)

and $x = 1 \Rightarrow y = 1^2 = 1$ which is pt P(1, 1)

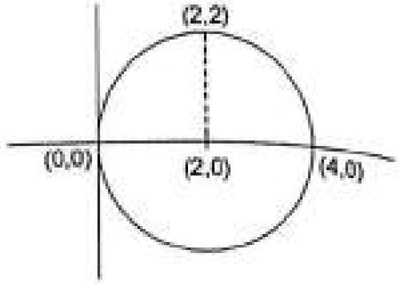
So required area is

$$= \int_0^1 x\,dx - \int_0^1 x^2 dx$$

$$\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Q.123 Consider an ant crawling along the curve $(x - 2)^2 + y^2 = 4$, where x and y are in meters. The ant starts at the point (4, 0) and moves counter-clockwise with a speed of 1.57 meters per second. The time taken by the ant to reach the point (2, 2) is (in seconds) ______.

[ME, GATE-2015 : 2 Marks, Set-1]



Solution: (2)

$(x-2)^2 + (y^2) = (2)^2$, is a circle of radius 2 m and centre at (2, 0)

Time to reach from (4, 0) to (2, 2) is

$$\text{time} = \frac{\text{Distance}}{\text{Speed}} = \frac{\left(\frac{2\pi r}{4}\right)}{1.57} = \frac{\left(\frac{2\pi 2}{4}\right)}{1.57} = \frac{\pi}{1.57} = 2 \text{ sec.}$$

Q.124 Consider a spatial curve in three-dimensional space given in parametric form by

$$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = \frac{2}{\pi}t,\ 0 \le t \le \frac{\pi}{2}$$

The length of the curve is ________. **[ME, GATE-2015 : 2 Marks, Set-1]**

Solution: (1.86)

$$S = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$S = \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\frac{2}{\pi}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{1+\left(\frac{4}{\pi^2}\right)}\, dt = \sqrt{1+\left(\frac{4}{\pi^2}\right)}\,[t]_0^{\pi/2} = \sqrt{1+\left(\frac{4}{\pi^2}\right)}\left(\frac{\pi}{2}\right) = 1.86$$

Q.125 The expression $V = \int_0^H \pi R^2 (1 - h/H)^2\, dh$ for the volume of a cone is equal to

(a) $\int_0^R \pi R^2 (1-h/H)^2\, dr$

(b) $\int_0^R \pi R^2 (1-h/H)^2\, dh$

(c) $\int_0^H 2\pi r H(1-r/R)\, dh$

(d) $\int_0^R \pi r H\left(1-\frac{r}{R}\right)^2 dr$

[EE, GATE-2006, 2 marks]

Solution: (d)

We consider options (a) and (d) only, because these contains variable r, as variable of integration.
By integrating (d), we get
$1/3\, \pi a^2 H$, which is volume of a cone.

Q.126 The line integral of function $F = yzi$, in the counter clockwise direction, along the circle $x^2 + y^2 = 1$ at $z = 1$ is

(a) -2π

(b) $-\pi$

(c) π

(d) 2π

[EE, 2014 : 2 Marks, Set-1]

Solution : (b)

$$\bar{F} = yz\,\hat{i}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 0 & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0-y) + \hat{k}(0-z) = 0\hat{i} + y\hat{j} - z\hat{k}$$

By Stokes theorem,

$$\int_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot\hat{n}ds = \int_S (y\hat{i}-z\hat{k})\cdot\hat{k}\, ds$$

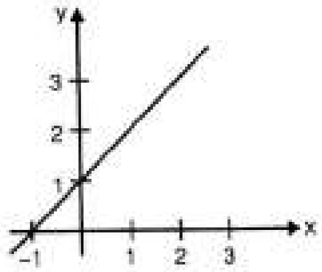
$$= \int_S -z\, ds \qquad \text{Since } z = 1$$

$$= \int_S -1ds = (-1)S = (-1)\pi = -\pi$$

where S is surface area of $x^2 + y^2 = 1$

$\therefore \quad S = \pi(1)^2 = \pi$

Q.127 The following plot shows a function y which varies linearly with x. The value of the integral $I = \int_1^2 y\, dx$ is



(a) 1.0
(b) 2.5
(c) 4.0
(d) 5.0 **[EC, GATE-2007, 1 mark]**

Solution: (b)

Equation of line with slope 1 and y – intercept of 1 is,

$$y = x + 1$$

$$I = \int_1^2 y\, dx = \int_1^2 (x+1)\, dx$$

$$= \left.\frac{(x+1)^2}{2}\right|_1^2 = \frac{1}{2}(9-4) = 2.5$$

Q.128 The value of the integral of the function $g(x,y) = 4x^3 + 10y^4$ along the straight line segment from the point (0,0) to the point (1,2) in the x-y plane is

(a) 33
(b) 35
(c) 40
(d) 56 **[EC, GATE-2008, 2 marks]**

Solution: (a)

Equation of straight line from point (0,0) to (1,2) is

$$y - 0 = \frac{(2-0)}{(1-0)}(x-0)$$

or

$$y = 2x$$

$$g(x, y) = 4x^3 + 10y^4$$

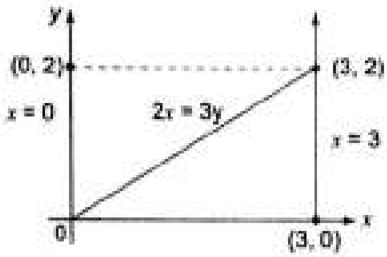
$$= 4x^3 + 10(2x)^4 = 4x^3 + 160x^4$$

$$\int_0^1 (4x^3 + 160x^4)dx = \left.\left(\frac{4x^4}{4} + \frac{160x^5}{5}\right)\right|_0^1 = 1 + 32 = 33$$

Q.129 A triangle in the xy-plane is bounded by the straight lines $2x = 3y$, $y = 0$ and $x = 3$. The volume above the triangle and under the plane $x + y + z = 6$ is ______.

[EC, 2016 : 2 Marks, Set-3]

Solution:



$$\text{Volume} = \iiint dz\,dx\,dy = \iint z\,dy\,dx$$

$$= \int_0^3 \int_0^{2/3x} (6 - x - y)\,dy\,dx = \int_0^3 \left(6y - xy - \frac{y^2}{2}\right)\Bigg|_0^{2/3x} dx$$

$$= \int_0^3 \left(4x - \frac{8}{9}x^2\right)dx = \left(4\frac{x^2}{2} - \frac{8}{9}\cdot\frac{x^3}{3}\right)\Bigg|_0^3$$

$$= \left[2x^2 - \frac{8}{9}\left(\frac{x^3}{3}\right)\right]_0^3 = 18 - 8 = 10$$

Q.130 The area between the parabola $x^2 = 8y$ and the straight line $y = 8$ is ______

[CE, 2016 : 2 Marks, Set-II]

Solution:

Parabola is $x^2 = 8y$

$$y = \frac{x^2}{8} \text{ and straight is } y = 0$$

At the point of intersection, we have

$$\frac{x^2}{8} = 8$$

$\Rightarrow$ $x = -8, 8$ and $y = 8 \geq y = \dfrac{x^2}{8}$

$\therefore$ Required area is $\int_{x=-8}^{8}\left(8 - \frac{x^2}{8}\right)dx$

$$= 2\int_0^8\left(8 - \frac{x^2}{8}\right)dx \left(\because 8 - \frac{x^2}{8} \text{ is even function}\right)$$

$$= 2\left[8x - \frac{x^3}{24}\right]_0^8 = \frac{256}{3} = 85.33 \text{ sq.units}$$

2.13 MULTIPLE INTEGRALS AND THEIR APPLICATIONS

1. Double integrals
2. Change of order of integration
3. Double integrals in polar co-ordinates
4. Areas enclosed by plane curves
5. Triple integrals

2.13.1 Double Integrals

The definite integral $\int_a^b f(x)dx$ is defined as the limit of the sum

$f(x_1)\delta x_1 + f(x_2)\delta x_2 + ... + f(x_n)\delta x_n$,

where $n \to \infty$ and each of the lengths δx_1, δx_2, ... tends to zero. A double integral is its counterpart in two dimensions.

Consider a function f(x, y) of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas δA_1, δA_2.........δA_n. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + ... + f(x_n, y_n)\delta A_n$, i.e. $\sum_{r=1}^{n} f(x_r, y_r)\delta A_r$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the double integral of f(x, y) over the region R and is written as $\iint_R f(x,y)dA$

Thus
$$\iint_R f(x,y)dA = \underset{\substack{n\to\infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \quad ...(i)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purposes of evaluation, (i) is expressed as the repeated integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dxdy$. Its value is found as follows:

1. When y_1, y_2 are functions of x and x_1, x_2 are constants, f(x, y) is first integrated w.r.t.y (keeping x fixed) between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2 i.e.

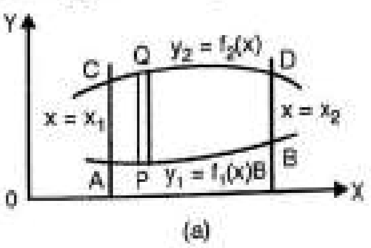
$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)dy}\, dx}$$

where integrations carried from the inner to the outer rectangle.

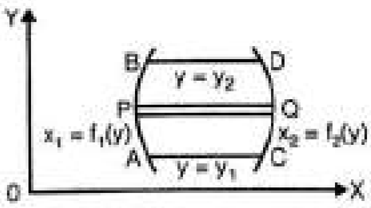
Fig. (a) below illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area ABDC.



(a)



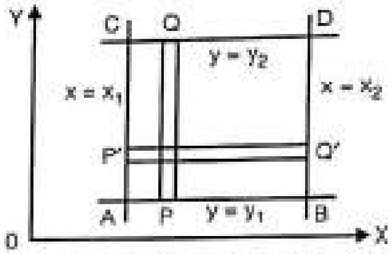
(b)

2. When x_1, x_2 are functions of y and y_1, y_2 are constants, f(x, y) is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. between the limits y_1, y_2, i.e.

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)dx}\, dy}$$ which is geometrically illustrated by Fig. (b) above

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy. Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area ABDC.



3. When both pairs of limits are constants, the region of integration is the rectangle ABDC (Fig.)
In I_1, we integrated along the vertical strip PQ and then slide it from AC to BD.
In I_2, we integrate along the horizontal strip P'Q' and then slide it fro AB to CD.
Here obviously $I_1 = I_2$.
Thus for constant limits, it hardly mattery whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

ILLUSTRATIVE EXAMPLES

Example:

Evaluate $\int_0^5 \int_0^{x^2} x(x+y^2)dxdy$.

Solution:

$$I = \int_0^5 dx \int_0^{x^2} (x^2 + xy^2)dy = \int_0^5 \left[x^2 y + x.\frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^5 \left[x^2.x^2 + x.\frac{x^6}{3}\right]dx = \int_0^5 \left(x^4 + \frac{x^7}{3}\right)dx = \left|\frac{x^5}{5} + \frac{x^8}{24}\right|_0^5$$

$$= \frac{5^5}{5} + \frac{5^8}{24} \cong 16901.04$$

2.13.2 Change of order of Integration

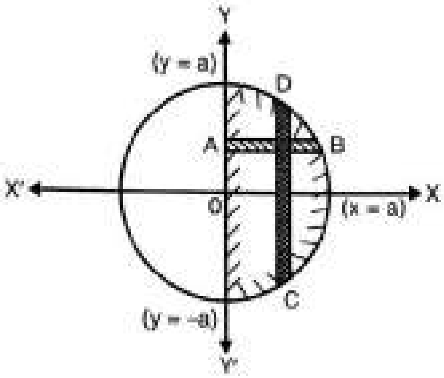
In a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketh of the region of integration.
The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

ILLUSTRATIVE EXAMPLES

Example: 1

Change the order of integration in the integral, $I = \int_{-a}^{a}\int_{0}^{\sqrt{(a^2-y^2)}} f(x,y)dxdy$.

Solution:



The elementary strip AB from x = 0 to $x = \sqrt{a^2 - y^2}$ (corresponding to the circle $x^2 + y^2 = a^2$), can be slided up from y = –a to y = a and integration is carried out. This shaded semi-circular area is, therefore, the region of integration (Figure above). This corresponds to the given integral

$I = \int_{-a}^{a}\int_{0}^{\sqrt{(a^2-y^2)}} f(x,y)dxdy$.

The order of integation can be changed, if we first integrate with respect to y along a vertical strip CD (going from $y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$), and then integrate with respect to x as x goes from x = 0 to x = a. (i.e. slide the strip CD from left to right from x = 0 to x = a)

This will result in the integral,

$$I = \int_{0}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)dy$$

or

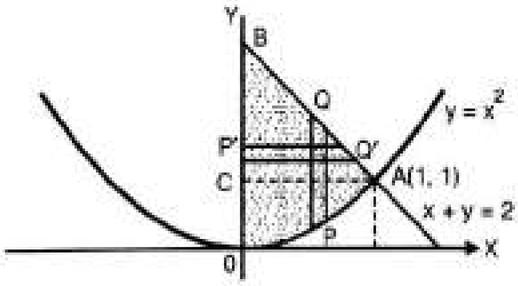
$$= \int_{0}^{a}\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)dy\,dx$$

Example: 2

Change the order of integration in $I = \int_{0}^{1}\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same.

Solution:

Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line y = 2 – x. Such a strip slides from x = 0 to x = 1, giving the region of integration as the curvilinear triangle OAB (shaded) in Figure.



On changing the order of integration, we first integrate w.r.t. x along a horizontal strip P'Q' and that requires the splitting up of the region OAB into two parts by the line AC (y= 1), i.e. the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from x = 0 to x = $\sqrt{y}$ and those for y are from y = 0 to y = 1. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from x = 0 to x = 2 − y and those for y are from y = 1 to y = 2. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx$$

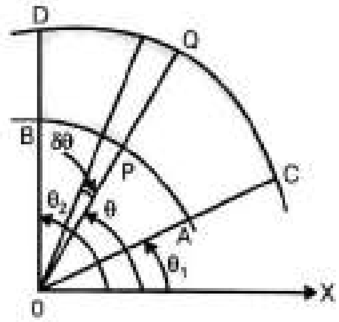
$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx + \int_1^2 dy \int_0^{2-y} xy\, dx$$

$$= \int_0^1 dy \left|\frac{x^2}{2}.y\right|_0^{\sqrt{y}} + \int_1^2 dy \left|\frac{x^2}{2}.y\right|_0^{2-y}$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

2.13.3 Double Integrals in Polar Co-ordinates

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta) dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.



Illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness δθ.

Then $\int_{r_1}^{r_2} f(r, \theta) dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area ACDB. The order of integration may be changed with appropriate changes in the limits.

ILLUSTRATIVE EXAMPLES

Example:

Calculate $\iint r^3 dr\, d\theta$ over the area included between the circles r = 2 sin θ and r = 4 sin θ.

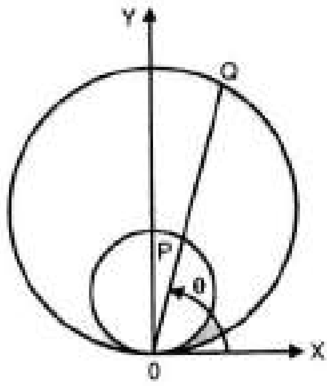
Solution:

Given circles $r = 2\sin\theta$ ... (i)

and $r = 4\sin\theta$ ... (ii)

are shown in Figure below. The shaded area between these circles is the region of integration. If we integrate first w.r.t. r, then its limits are from P(r = 2 sin θ) to Q(r = 4 sin θ) and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr$$



$$= \int_0^{\pi} d\theta \left|\frac{r^4}{4}\right|_{2\sin\theta}^{4\sin\theta}$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta$$

$$= 60 \times 2\int_0^{\pi/2} \sin^4\theta\, d\theta$$

using reduction formula,

$$\int_0^{\pi/2} \sin^4\theta\, d\theta = \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots}\cdot\left(\frac{\pi}{2}\right)$$

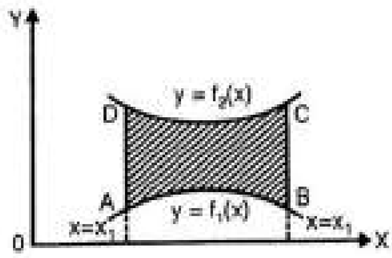
Here n = 4 [using walle's formula with n is even]

So, $$\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\times1}{4\times2}\left(\frac{\pi}{2}\right)$$

So the required integral, $$I = 120 \times \frac{3\times1}{4\times2}\left(\frac{\pi}{2}\right) = 22.5\,\pi$$

2.13.4 Area Enclosed by Plane Curves

The area enclosed by curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ is shown in figure below and is given by the double integral $\int_{y_2}^{y_1}\int_{f_1(y)}^{f_2(y)} dx\, dy$.



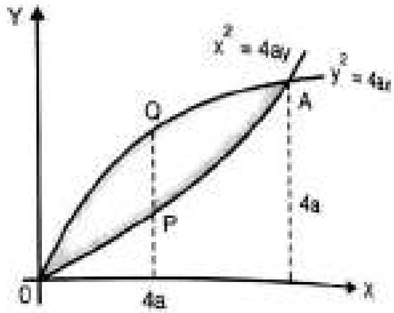
ILLUSTRATIVE EXAMPLES

Example:

Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Solution:

The equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O(0, 0) and A(4a, 4a). As such for the shaded area between these parabolas (Fig. below) x varies from 0 to 4a and y varies from P to Q i.e. from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the required area



$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dy\,dx$$

$$= \int_0^{4a}(2\sqrt{(ax)} - x^2/4a)dx$$

$$= \left|2\sqrt{a}\cdot\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a}$$

$$= \frac{32}{2}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.131 The value of $\int_0^3\int_0^x(6-x-y)dx\,dy$ is

(a) 13.5
(b) 27.0
(c) 40.5
(d) 54.0

[CE, GATE-2008, 2 marks]

Solution: (a)

$$\int_0^3\int_0^x(6-x-y)dxdy = \int_0^3\left[(6-x)y - \frac{y^2}{2}\right]_0^x dx = \int_0^3\left[(6-x)x - \frac{x^2}{2}\right]dx = 13.5$$

Q.132 The volume of an object expressed in spherical co-ordinates is given by $V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2\sin\phi\,dr\,d\phi\,d\theta$. The value of the intergal is

(a) $\frac{\pi}{3}$
(b) $\frac{\pi}{6}$
(c) $\frac{2\pi}{3}$
(d) $\frac{\pi}{4}$

[ME, GATE-2004, 2 marks]

Solution: (a)

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2\sin\phi\cdot dr\cdot d\phi\cdot d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{r^3}{3}\right]_0^1 \sin\phi\,d\phi\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}[-\cos\phi]_0^{\pi/3}d\theta = \frac{1}{3}\times\frac{1}{2}\times\int_0^{2\pi}d\theta = \frac{1}{3}\times\frac{1}{2}\times 2\pi = \frac{\pi}{3}$$

Q.133 Changing the order of the integration in the double integral $I=\int_0^8\int_{x/4}^2 f(x, y)dydx$ leads to $I=\int_r^s\int_p^q f(x, y)dxdy$. What is q?

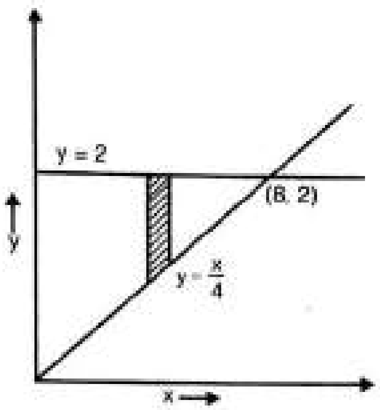
(a) 4 y
(b) $16y^2$
(c) x
(d) 8

[ME, GATE-2005, 1 mark]

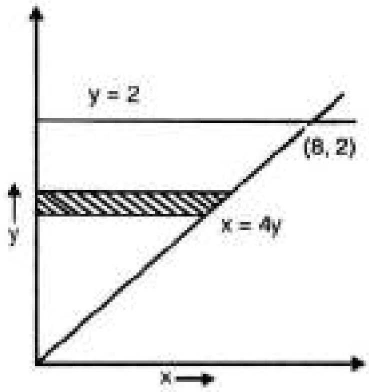
Solution: (a)

When
$$I = \int_0^8\int_{x/4}^2 f(x\cdot y)dydx$$

i.e.



Now,



$$I = \int_0^2\int_0^{4y} f(x,y)dxdy$$

$\therefore$
$$q = 4y$$

Q.134 By a change of variable x (u, v) = uv, y (u, v) = v/u is double integral, the integrand f(x, y) changes to f(uv, v/u) ϕ (u,v). Then, ϕ (u, v) is

(a) 2 u/v
(b) 2 uv
(c) v^2
(d) 1

[ME, GATE-2005, 2 marks]

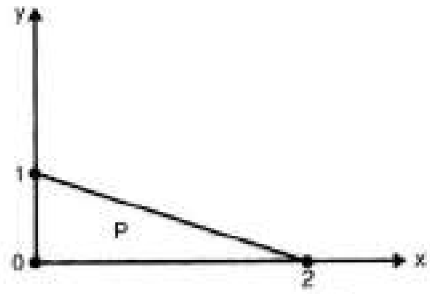
Solution: (a)

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

and
$$\frac{\partial y}{\partial u} = -\frac{v}{u^2} \qquad \frac{\partial y}{\partial v} = \frac{1}{u}$$

and $$\phi(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}$$

Q.135 Consider the shaded triangular region P shown in the figure. What is $\iint_P xy\,dx\,dy$?

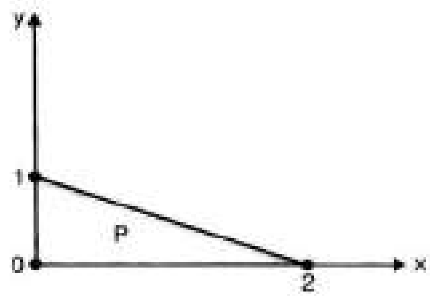


(a) $\frac{1}{6}$ (b) $\frac{2}{9}$

(c) $\frac{7}{16}$ (d) 1

[ME, GATE-2008, 2 marks]

Solution: (a)



The equation of the st line with x-intercept = 2 and y-intercept = 1 is

$$\frac{x}{2} + \frac{y}{1} = 1$$

$$\Rightarrow \quad y = 1 - \frac{x}{2}$$

$$\Rightarrow \quad x = 2 - 2y$$

$$\int_0^1 \int_0^{(2-2y)} (xy\,dx)dy = \int_0^1 \left[\left(\frac{yx^2}{2} \right)\Big|_0^{2-2y} \right] dy$$

$$= \int_0^1 \frac{y}{2}(2-2y)^2\,dy = \int_0^1 2y(1-y)^2\,dy = \frac{1}{6}$$

Alternatively, we may also write this integral as $\int_0^2 \int_0^{\frac{2-x}{2}} (xy\,dy)dx$ which is also $= \frac{1}{6}$

Q.136 The value of the integral $\int_0^2\int_0^x e^{x+y}dy\,dx$

(a) $\frac{1}{2}(e-1)$

(b) $\frac{1}{2}(e^2-1)^2$

(c) $\frac{1}{2}(e^2-e)$

(d) $\frac{1}{2}\left(e-\frac{1}{e}\right)^2$

[ME, GATE-2014 : 2 Marks, Set-2]

Solution : (b)

$$I = \int_0^2\left[\int_0^x e^{x+y}dy\right]dx$$

$$\int_0^x e^{x+y}dy = e^{x+y}dy\Big|_0^x = e^{2x}-e^x$$

$$I = \int_0^2 (e^{2x}-e^x)dx$$

$$I = \frac{1}{2}e^{2x}\Big|_0^2 - e^x\Big|_0^2 = \frac{1}{2}(e^4-1)-(e^2-1)$$

$$I = \frac{1}{2}e^4-\frac{1}{2}-e^2+1 = \frac{1}{2}e^4-e^2+\frac{1}{2} = \frac{1}{2}(e^4-2e^2+1)$$

$$I = \frac{1}{2}(e^2-1)^2$$

Q.137 A surface S(x, y) = 2x + 5y − 3 is integrated once over a path consisting of the points that satisfy $(x+1)^2+(y-1)^2=\sqrt{2}$. The integral evaluates to

(a) $17\sqrt{2}$

(b) $\frac{17}{\sqrt{2}}$

(c) $\frac{\sqrt{2}}{17}$

(d) 0

[EE, GATE-2006, 2 marks]

Solution: (d)

$$x+1 = \sqrt{2}\cos\theta \quad ; \quad y-1 = \sqrt{2}\sin\theta$$

$$x = \sqrt{2}\cos\theta-1 \; ; \; y = \sqrt{2}\sin\theta+1$$

$$= \int_0^{2\pi}(2\sqrt{2}\cos\theta-2+5\sqrt{2}\sin\theta+5-3)d\theta$$

$$= \int_0^{2\pi}(2\sqrt{2}\cos\theta+5\sqrt{2}\sin\theta)d\theta$$

$$= 2\sqrt{2}(\sin\theta)\Big|_0^{2\pi}+5\sqrt{2}(-\cos\theta)\Big|_0^{2\pi}$$

$$= 2\sqrt{2}\,(\sin 2\pi-\sin 0)-5\sqrt{2}(\cos 2\pi-\cos 0)$$

$$= 2\sqrt{2}(0-0)-5\sqrt{2}(1-1)=0$$

Q.138 f(x, y) is a continuous function defined over $(x, y) \in [0, 1] \times [0, 1]$. Given the two constraints, $x > y^2$ and $y > x^2$, the volume under f(x, y) is

(a) $\int_{y=0}^{y=1}\int_{x=y^2}^{x=\sqrt{y}} f(x,y)dxdy$

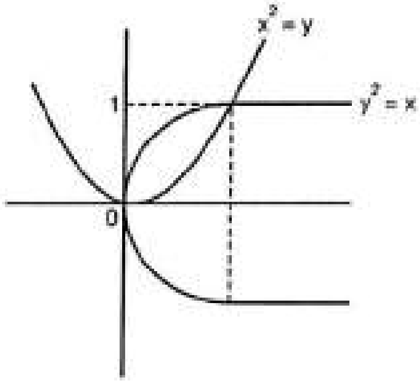
(b) $\int_{y=x^2}^{y=1}\int_{x=y^2}^{x=1} f(x,y)dxdy$

(c) $\int_{y=0}^{y=1}\int_{x=0}^{x=1} f(x,y)dxdy$

(d) $\int_{y=0}^{y=\sqrt{x}}\int_{x=0}^{x=\sqrt{y}} f(x,y)dxdy$

[EE, GATE-2009, 2 marks]

Solution: (a)



$$\int_{y=0}^{y=1}\int_{x=y^2}^{x=y} f(x, y)dx\,dy$$

Q.139 To evaluate the double integral $\int_0^8\left(\int_{y/2}^{(y/2)+1}\left(\frac{2x-y}{2}\right)dx\right)dy$, we make the substitution = $\left(\frac{2x-y}{2}\right)$ and $v = \frac{y}{2}$. The integral will reduce to

(a) $\int_0^4\left(\int_0^2 2u\,du\right)dv$

(b) $\int_0^4\left(\int_0^1 2u\,du\right)dv$

(c) $\int_0^4\left(\int_0^1 u\,du\right)dv$

(d) $\int_0^4\left(\int_0^2 u\,du\right)dv$

[EE, GATE-2014 : 2 Marks, Set-2]

Solution : (b)

$$\int_0^8\left(\int_{y/2}^{(y/2)+1}\left(\frac{2x-y}{2}\right)dx\right)dy$$

$$\frac{2x-y}{2} = u$$

$$x - \frac{y}{2} = u$$

$$dx = du$$

$$\text{at } x = \frac{y}{2}$$

$$u = \frac{\frac{2y}{2} - y}{2} = 0$$

$$\text{at} \quad x = \frac{y}{2} + 1$$

$$u = \frac{2\left(\frac{y}{2}+1\right) - y}{2} = \frac{y+2-y}{2} = 1$$

Thus, itegral becomes $\int_0^8 \left[\int_0^1 u\,du\right] dy$

$$v = \frac{y}{2}$$

$$dv = \frac{dy}{2} \qquad \Rightarrow \quad dy = 2\,dv$$

$\Rightarrow \quad v = 0$ $\qquad y = 8$

$\Rightarrow \quad v = 4$

$$= \int_0^4 \left[\int_0^1 u\,du\right] \times 2dv = \int_0^4 \left[\int_0^1 2u\,du\right] dv$$

Q.140 The volume enclosed by the surface $f(x, y) = e^x$ over the triangle bounded by the lines $x = y$; $x = 0$; $y = 1$ in the xy plane is ______.

[EE, GATE-2015 : 2 Marks, Set-2]

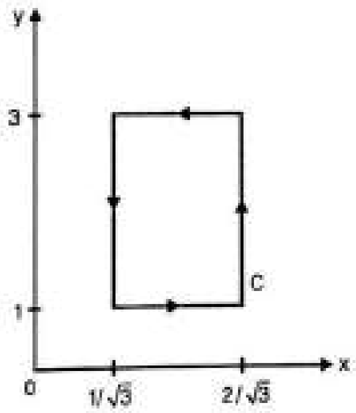
Solution: (0.71828)

$$\text{Volume} = \iint f(x,y)\,dx\,dy = \int_0^1 \int_0^y e^x\,dx\,dy$$

$$= \int_0^1 [e^x]_0^y\,dy = \int_0^1 (e^y - 1)\,dy = (e^y - y)\Big|_0^1 = (e-1) - (1-0)$$

$$= e - 1 - 1 = e - 2 = 0.71828$$

Q.141 If $\vec{A} = xy\hat{a}_x + x^2\hat{a}_y$, $\oint_C \vec{A} \cdot d\vec{l}$ over the path shown in the figure is



(a) 0

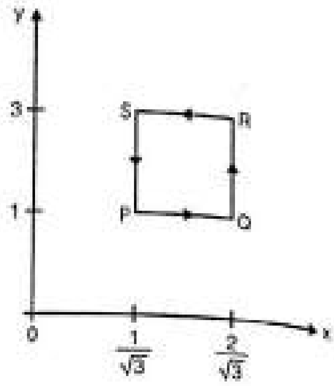
(b) $\frac{2}{\sqrt{3}}$

(c) 1

(d) $2\sqrt{3}$

[EC, GATE-2010, 2 marks]

Solution: (c)



$$\vec{A} = xy\hat{a}_x + x^2\hat{a}_y$$
$$\vec{l} = x\hat{a}_x + y\hat{a}_y$$
$$\vec{dl} = dx\hat{a}_x + dy\hat{a}_y$$
$$\vec{A}.\vec{dl} = xy\,dx + x^2\,dy$$

P – Q: $y = 1 \;,\; dy = 0$

$$\int_P^Q \vec{A}.\vec{dl} = \int_{1/\sqrt{3}}^{2/\sqrt{3}} x\,dx = \frac{x^2}{2}\Big|_{1/\sqrt{3}}^{2/\sqrt{3}} = \frac{1}{2}$$

Q – R: $x = \frac{2}{\sqrt{3}} \;,\; dx = 0$

$$\int_Q^R \vec{A}.\vec{dl} = \int_1^3 \left(\frac{2}{\sqrt{3}}\right)^2 dy = \frac{4}{3}(3-1) = \frac{8}{3}$$

R – S: $y = 3 \;,\; dy = 0$

$$\int_R^S \vec{A}.\vec{dl} = \int_{2/\sqrt{3}}^{1/\sqrt{3}} 3x\,dx = \frac{3}{2}x^2\Big|_{2/\sqrt{3}}^{1/\sqrt{3}} = \frac{3}{2}\left(\frac{1}{3}-\frac{4}{3}\right) = \frac{-3}{2}$$

S – P: $x = \frac{1}{\sqrt{3}} \;,\; dx = 0$

$$\int_S^P \vec{A}.\vec{dl} = \int_3^1 \left(\frac{1}{\sqrt{3}}\right)^2 dy = \frac{1}{3}(1-3) = \frac{-2}{3}$$

So,

$$\oint_C \vec{A}.\vec{dl} = \int_P^Q \vec{A}.\vec{dl} + \int_Q^R \vec{A}.\vec{dl} + \int_R^S \vec{A}.\vec{dl} + \int_S^P \vec{A}.\vec{dl}$$
$$= \frac{1}{2} + \frac{8}{3} - \frac{3}{2} - \frac{2}{3} = 1$$

Q.142 The double integral $\int_0^a\int_0^y f(x,y)\,dx\,dy$ is equivalent to

(a) $\int_0^x\int_0^y f(x,y)\,dx\,dy$

(b) $\int_0^a\int_x^y f(x,y)\,dx\,dy$

(c) $\int_0^a\int_x^a f(x,y)\,dy\,dx$

(d) $\int_0^a\int_0^a f(x,y)\,dx\,dy$

[IN, GATE-2015 : 1 Mark]

Solution: (c)

$$I = \int_0^a\int_0^y f(x,y)\,dx\,dy$$

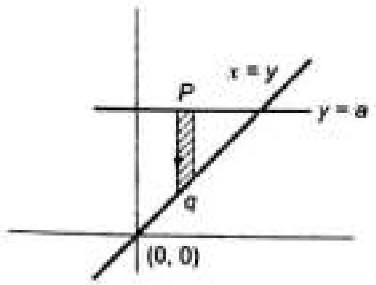
Limit of x:
Lower limt $x = 0$
Upper limt $x = y$

Limit of y:

Lower limt $y = 0$

Upper limt $y = a$

By change of order of integration limt of y



Limit of y:

Lower limt $y = x$

Upper limt $y = a$

Limt of x:

Lower limit $x = 0$

Upper limt $x = a$

So. $I = \int_0^a \int_x^a f(x, y)\, dy\, dx$

Q.143 The integral $\frac{1}{2\pi}\iint_D (x + y + 10)dx\,dy$, where D denotes the disc: $x^2 + y^2 \le 4$, evaluates to ____.

[EC, 2016 : 2 Marks, Set-1]

Solution:

Put

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dxdy = rdrd\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^2 (r(\cos\theta + \sin\theta) + 10)r\,drd\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^2 (r^2(\cos\theta + \sin\theta) + 10r)\,drd\theta$$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi}(\cos\theta + \sin\theta)\left(\frac{r^3}{3}\right)\Bigg|_0^2 d\theta + 10\int_0^{2\pi}\left(\frac{r^2}{2}\right)\Bigg|_0^2 d\theta\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{8}{3}(\cos\theta + \sin\theta)d\theta + \frac{1}{2\pi}\int_0^{2\pi} 5\cdot(4)d\theta$$

$$= \frac{1}{2\pi}\left[\frac{8}{3}(\sin\theta - \cos\theta)\right]_0^{2\pi} + \frac{1}{2\pi}\cdot 20(2\pi)$$

$$= \frac{1}{2\pi}\left(\frac{8}{3}(0-1) - (0-1) + 20\right) = 0 + 20 = 20$$

Q.144 Suppose C is the closed curve defined as the circle $x^2 + y^2 = 1$ with C oriented anti clockwise. The value of $\oint(xy^2dx + x^2ydy)$ over the curve C equals ______.

[EC, 2016 : 2 Marks, Set-2]

Solution:

By Green's theorem

$$\int xy^2dx + x^2ydy = \iint_R\left(\frac{d}{dx}(x^2y) - \frac{d}{dy}(xy^2)\right)dxdy = \iint_R(2xy - 2xy) = 0$$

Q.145 The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is

(a) $\frac{59}{6}$
(b) $\frac{9}{2}$
(c) $\frac{10}{3}$
(d) $\frac{7}{6}$

[CE, 2016 : 2 Marks, Set-I]

Solution: (b)

At the point of intersection of the curves,
$y = x^2 + 1$ and $x + y = 3$ i.e., $y = 3 - x$, we have

$$x^2 + 1 = 3 - x$$

$\Rightarrow$ $$x^2 + x - 2 = 0$$

$\Rightarrow$ $x = -2, 1$ and $3 - x \geq x^2 + 1$

$\therefore$ Required area is $\iint_R dydx$

$$= \int_{x=-2}^{1}\left[\int_{y=x^2+1}^{3-x} dy\right]dx = \int_{-2}^{1}\{3-x\}-\left(x^2+1\right)dx$$

$$= \left(\frac{-x^3}{3}-\frac{x^2}{2}+2x\right)_{-2}^{1} = \frac{9}{2}$$

2.13.5 Triple Integrals

Consider a function f(x, y, z) defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r,$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the triple integral of f(x, y, z) over the region V and is denoted by

$$\iiint f(x,y,z)dV$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dx\,dy\,dz\,.$$

If x_1, x_2 are constants; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows:
First f(x, y, z) is integrated w.r.t. z between the limits, z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus
$$I = \boxed{\int_{x_1}^{x_2}\boxed{\int_{y_1(x)}^{y_2(x)}\boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz}dy}dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.
The order of integration may be different for different types of limits.

ILLUSTRATIVE EXAMPLES

Example: 1

Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz$

Solution:

Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1}\int_{0}^{z}\left|xy+\frac{y^2}{2}+yz\right|_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(2z)+\frac{1}{2}4xz\right]dx\,dz = 2\int_{-1}^{1}\left|\frac{x^2z}{2}+z^2x+\frac{x^2}{2}z\right|dz$$

$$= 2\int_{-1}^{1}\left(\frac{z^2}{2}+z^3+\frac{z^3}{2}\right)dz = 4\left|\frac{z^4}{4}\right|_{-1}^{1} = 0$$

Example: 2

Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz$.

Solution:

We have, $$I = \int_0^1 x\left[\int_0^{\sqrt{(1-x^2)}} y\left\{\int_0^{\sqrt{(1-x^2-y^2)}} z\,dz\right\}dy\right]dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y\cdot\left|\frac{z^2}{2}\right|_0^{\sqrt{(1-x^2-y^2)}} dy\right\}dx = \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y\cdot\frac{1}{2}(1-x^2-y^2)dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left|(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right|_0^{\sqrt{(1-x^2)}} dx = \frac{1}{8}\int_0^1[(1-x^2)^2\cdot 2x-(1-x^2)^2\cdot x]dx$$

$$= \frac{1}{8}\int_0^1(x-2x^3+x^5)dx = \frac{1}{8}\left|\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right|_0^1 = \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{48}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.146 The region specified by $\{(\rho, \phi, z): 3 \le \rho \le 5, \frac{\pi}{8} \le \varphi \le \frac{\pi}{4}, 3 \le z \le 4.5\}$ in cylindrical coordinates has volume of _____.

[EC, 2016 : 2 Marks, Set-1]

Solution:

$$V = \int_{\rho=3}^{5}\int_{\phi=\frac{\pi}{8}}^{\pi/4}\int_{z=3}^{4.5}\rho d\rho\, d\phi dz = \int_3^{4.5}\int_{\pi/8}^{\pi/4}\left(\frac{\rho^2}{2}\right)_3^5 d\phi dz = \int_3^{4.5}\int_{\pi/8}^{\pi/4} 8\cdot d\phi dz = 8\phi|_{\pi/8}^{\pi/4}\cdot z|_3^{4.5}$$

$$= 8\left(\frac{\pi}{4}-\frac{\pi}{8}\right)(4.5-3) = 8\cdot\frac{\pi}{8}\cdot(1.5) = 4.712$$

2.14 VECTORS

2.14.1 Introduction

This chapter deals with vectors and vector functions in 3-space and extends the differential calculus to these vector functions. Forces, velocities and various other quantities are vectors. This makes the algebra and calculus of these vector functions the natural instrument for the engineer and physicist in solid mechanics, fluid flow, heat flow, electrostatics, and so on. The engineer must understand these fields as the basis of the design and construction of system, or robots. In three dimensions (as opposed to higher dimensions), geometrical ideas become influential, enriching the theory, and many geometrical quantities (tangents and normals, for example) can be given by vectors.

We first explain the basic algebraic operations with vectors in 3-space. Vector differential calculus begins next with a discussion of vector functions, which represent vector fields and have various physical and geometrical applications. Then the basic concepts of differential calculus are extended to vector functions in a simple and natural fashion. Vector functions are useful in studying curves and their applications as paths of moving bodies in mechanics.

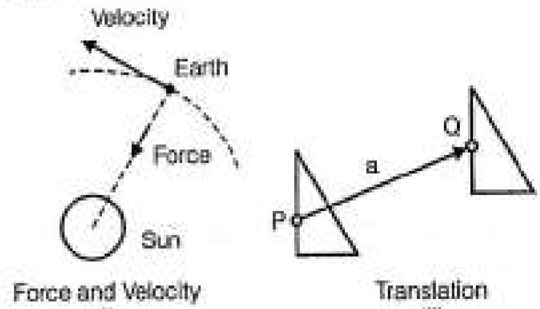
We finally discuss three physically and geometrically important concepts related to scalar and vector fields, namely, the gradient, divergence, and curl. Integral theorems involving these concepts follow in vector integral calculus.

2.14.2 Basic Definitions

In geometry and physics and its engineering applications we use two kinds of quantities, scalars and vectors. A scalar is a quantity that is determined by its magnitude, its number of units measured on a suitable scale. For instance, length, temperature, and voltage are scalars.

A vector is a quantity that is determined by both its magnitude and its direction; thus it is an arrow or directed line segment. For instance, a force is a vector, and so is a velocity, giving the speed and direction of motion (Figure below). We denote vectors by lower case bold face letters **a**, **b**, **v** etc.

A vector (arrow) has a tail, called its initial point, and a tip, called its terminal point. For instance, in Figure below, the triangle is translated (displaced without rotation); the initial point P of the vector **a** is the original position of a point and the terminal point Q is its position after the translation.



Force and Velocity (i) | Translation (ii)

The length (or magnitude) of a vector **a** (length of the arrow) is also called the norm (or Euclidean norm) of **a** and is denoted by |**a**|.

A vector of length 1 is called a unit vector.

2.14.3 Equality of Vectors

By definition, two vectors **a** and **b** are equal, written, **a** = **b**, if they have the same length and the same direction (Figure below). Hence a vector can be arbitrarily translated, that is, its initial point can be chosen arbitrarily. This definition is practical in connection with forces and other applications.

Equal vectors, a = b — Vectors having the same length but different direction — Vectors having the same direction but different length — Vectors having different length and different direction

Vectors

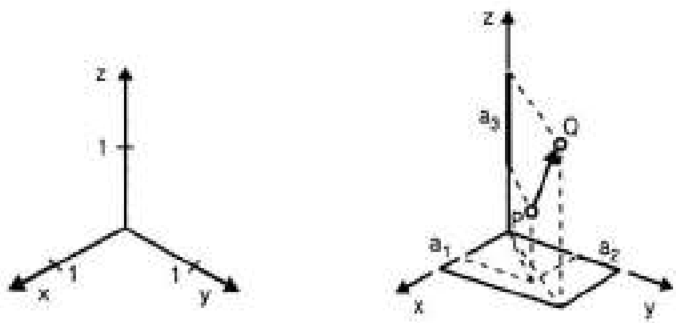
2.14.4 Components of a Vector

We choose an xyz Cartesian coordinate system in space, that is, a usual rectangular coordinate system with the same scale of measurement on the three mutually perpendicular coordinate axes. Then if a given vector **a** has initial point P: (x_1, y_1, z_1) and terminal point Q: (x_2, y_2, z_2) the three numbers,

1. $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$; are called the components of the vector a with respect to that coordinate system, and we write simply $a = [a_1, a_2, a_3]$.

   Length in Terms of Components: By definition, the length |a| of a vector a is the distance between its initial point P and terminal point Q. From the Pythagorean theorem, and figure (ii) below we see that

2. $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$



Cartesian coordinate (i) — Components of a vector (ii)

ILLUSTRATIVE EXAMPLES

Example:

Components and length of a vector.

The vector a with initial point P: (4, 0, 2) and terminal point Q; (6, – 1, 2) has the components $a_1 = 6 - 4 = 2$, $a_2 = -1 - 0 = -1$, $a_3 = 2 - 2 = 0$.

Solution:

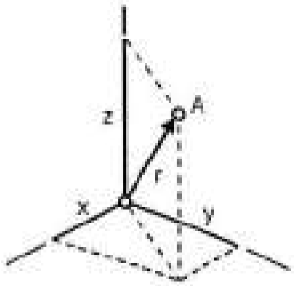
Hence,
$$\mathbf{a} = [2, -1, 0].$$
$$|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

If w choose (–1, 5, 8) as the initial point of a, the corresponding terminal point is (1, 4, 8).

If we choose the origin (0, 0, 0) as the initial point of a, the corresponding terminal point is (2, –1, 0); i.e. its coordinates equal the components of a, if origin is closer as initial point. This suggests that we can determine each point in space by a vector, as follows:

2.14.5 Position Vector

A Cartesian coordinate system being given, the position vector r of a point A; (x, y, z) is the vector with the vector with the origin (0, 0, 0) as the initial point and A as the terminal point. Thus, r = [x, y, z]. Furthermore, if we translate a vector **a**, with initial point P and terminal point Q, then corresponding coordinates of P and Q change by the same amount, so that the components of the vector remain unchanged. This proves



Position vector r of a point A: (x, y, z)

2.14.5.1 Vectors as Ordered Triples of Real Numbers

Theorem: A fixed Cartesian coordinate system being given, each vector is uniquely determined by its ordered triple of corresponding components. Conversely, to each ordered triple of real numbers $(a_1, a_2\ a_3)$ there corresponds precisely one vector $\mathbf{a} = [a_1, a_2, a_3]$. In particular, the ordered triple (0, 0, 0) corresponds to the zero vector "0", which has length 0 and no direction.

Hence a vector equation **a** = **b** is equivalent to the three equations $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$ for the components.

We see that from our "geometrical" definition of vectors as arrows we have arrived at an "algebraic" characterization by above Theorem. We could have started from the latter and reversed our process. This shows that the two approaches (i.e. "geometrical" and "algebraic" approaches) are equivalent.

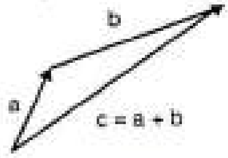
2.14.6 Vector Addition, Scalar Multiplication

Applications have suggested algebraic calculations with vectors thet are practically useful and almost as simple as calculations with numbers.

2.14.6.1 Definition: 1

Addition of Vectors: The sum **a** + **b** of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is obtained by adding

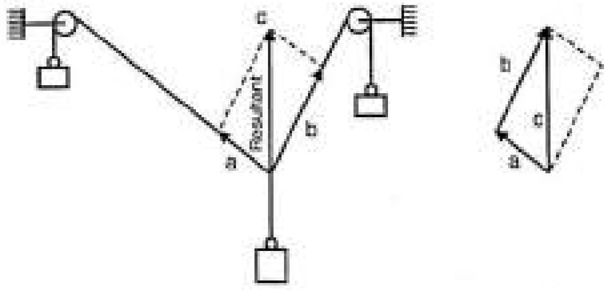
$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$



Vector addition

Geometrically, place the vectors as in Fig. above (the initial point of b at the terminal point of a): then **a** + **b** is the vector drawn from the initial point of **a** to the terminal point of **b**.

Figure shows that for forces, this addition is the parallelogram law by which we obtain the resultant of two forces in mechanics.



Resultant of two forces (parallgram law)

Figure illustrates (for the plane) that the "algebraic" way and the "geometric" way of vector addition amount to the same thing.

Basic properties of Vector addition follow immediately from the familiar laws for real numbers

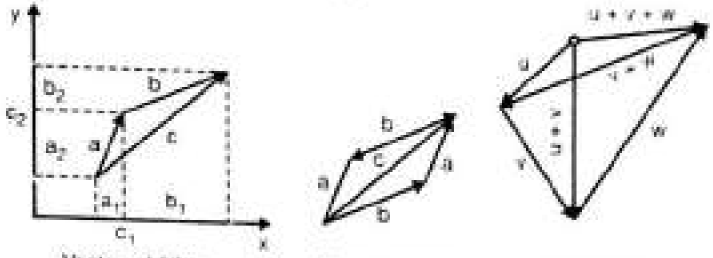
(a) $\dot{a} + \dot{b} = \dot{b} + \dot{a}$ (commutativity)

(b) $(\dot{u} + \dot{v}) + \dot{w} = \dot{u} + (\dot{v} + \dot{w})$ (associativity)

(c) $\dot{a} + 0 = 0 + \dot{a} = \dot{a}$

(d) $\dot{a} + (-\dot{a}) = 0$

where $-a$ denotes the vector having the length $|a|$ and the direction opposite to that of a.



Vector addition Commutativity of vector addition Associativity of vector addition

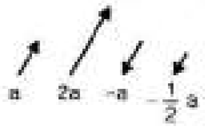
In property (b) above, instead of $u + (v + w)$ or $(u + v) + w$, we may simply write $u + v + w$ without brackets, and similarly for sums of more than three vectors. Also instead of $a + a$ we also write $2a$, and so on. This (and the notation $-a$ before) suggests that we define the second algebraic operation for vectors, namely, the multiplication of vectors by a scalar as follows.

2.14.6.2 Definition: 2

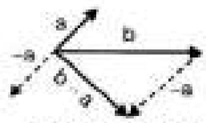
Scalar Multiplication (Multiplication by a Number): The product $c\dot{a}$ of any vector $a = [a_1, a_2, a_3]$ and any scalar c (real number c) is the vector obtained by multiplying each component of a by c,

$$ca = [ca_1, ca_2, ca_3]$$

Geometrically, if $a \neq 0$, then ca with $c > 0$ has the direction of $\dot{a}$ and with $c < 0$ the direction opposite to a. In any case, the length of ca is $|ca| = |c||a|$, and $ca = 0$ if $a = 0$ or $c = 0$ (or both).



Scalar multiplication [multiplication of vector by scalars (numbers)]



Difference of vectors

ILLUSTRATIVE EXAMPLES

Example:

Vector Addition and Multiplication by Scalars.

With respect to a given coordinate system, let

$$\dot{a} = [4, 0, 1] \text{ and } \dot{b} = \left[2, -5, \tfrac{1}{3}\right]$$

Solution:

Then $$-a = [-4, 0, -1],\ 7a = [28, 0, 7].\ a + b = \left[6, -5, \frac{4}{3}\right], \text{ and}$$

$$2(a - b) = 2\left[2, -5, \tfrac{2}{3}\right] = \left[4, 10, \tfrac{4}{3}\right] = 2\dot{a} - 2\dot{b}.$$

2.14.7 Unit Vectors

Any vector whose length is 1 is a unit vector i, j and k are example of special unit vectors, which are along x, y and z coordinate axes.

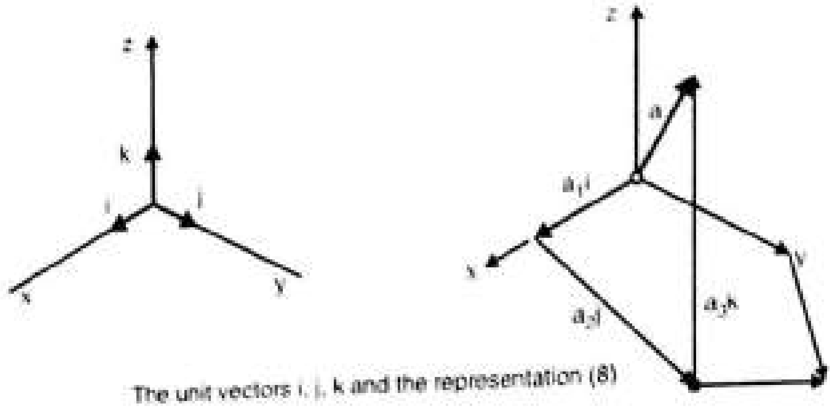
$$|i| = |j| = |k| = 1$$
$$u = \cos\theta\, i + \sin\theta\, j$$

gives every unit vector in the plane.

2.14.7.1 Representation of Vectors in Terms of i, j, and k

$$\dot{a} = [a_1, a_2, a_3] = a_1 i + a_2 j + a_3 k$$

In this representation, i, j, k are the unit vectors in the positive directions of the axes of a Cartesian coordinate system.



The unit vectors i, j, k and the representation (8)

i = [1, 0, 0] j = [0, 1, 0], k = [0, 0, 1]

and the right side of $a = a_1 i + a_2 j + a_3 k$ is a sum of three vectors parallel to the three axes.

ILLUSTRATIVE EXAMPLES

Example:

i, j, k Notation for Vectors:

Solution:

In previous example where a = [4, 0, 1] and $b = \left[2, -5, \frac{1}{3}\right]$,

we have $a = 4i + k$, $b = 2i - 5j + \frac{1}{3}k$, and so on, in i, j, k notation.

2.14.8 Length and Direction of Vectors

Any vector **a** may be written as a product of its length and direction as follows:

$$\dot{a} = |\dot{a}|\left(\frac{\dot{a}}{|\dot{a}|}\right)$$

here $|\dot{a}|$ is the length of vector and $\frac{\dot{a}}{|\dot{a}|}$ is a unit vector in direction of a

ILLUSTRATIVE EXAMPLES

Example: 1

Express 3i – 4j as a product of length and direction:

$$v = 3i - 4i$$

Solution:

$$\text{length of } v = |v| = \sqrt{3^2 + 4^2}$$

The unit vector in direction of $v = \frac{v}{|v|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$

$$\therefore \quad v = 3i - 4j = 5\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

Note that $\left|\frac{3}{5}i - \frac{4}{5}j\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

Since, $\frac{3}{5}i - \frac{4}{5}j$ is a unit vector.

Example: 2

Find a unit vector in direction of 4i + 6j.

Solution:

The required vector is $\frac{v}{|v|} = \frac{4i + 6j}{\sqrt{4^2 + 6^2}} = \frac{4}{\sqrt{52}}i + \frac{6}{\sqrt{52}}j$.

Example: 3

Find unit vector, tangent and normal to the curve

$$y = \frac{x^3}{2} + \frac{1}{2} \text{ at pt}(1,1)$$

Solution:

Unit vector tangent to curve:

$$y' = \left[\frac{3x^2}{2}\right]_{(1,1)} = \frac{3.1^2}{2} = \frac{3}{2}$$

Any vector with slope of $\frac{3}{2}$ can be written as

$$v = k(2i + 3j)$$

$$|v| = k\sqrt{2^2 + 3^2} = \sqrt{13}\,A$$

A unit vector in direction of v is

$$u = \frac{v}{|v|} = \frac{k(2i + 3j)}{\sqrt{13}\,k} = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{3}}j$$

Note also that $-u = \frac{-2}{\sqrt{13}}i - \frac{3}{\sqrt{13}}j$

is another unit vector tangent to the curve, but in opposite direction to u.

Unit vector normal to curve:

$$u = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j$$

Any vector normal to ai + bj is of the form of bi – aj, since product of this slopes is

$$\left(\frac{b}{a}\right)\left(-\frac{a}{b}\right) = -1$$

So a vector normal to $u = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j$ is $n = \frac{3}{\sqrt{13}}i - \frac{2}{\sqrt{13}}j$

Note that $-n = -\frac{3}{\sqrt{13}}i + \frac{2}{\sqrt{13}}j$ is another unit vector normal to the curve, but in opposite direction to n.

ILLUSTRATIVE EXAMPLES FROM GATE

Q147 Consider the time-varying vector

$I = \hat{x}15\cos(\omega t) + \hat{y}5\sin(\omega t)$ in Cartesian coordinates, where $\omega > 0$ is a constant. When the vector magnitude $|I|$ is at its minimum value, the angle θ that I makes with the x axis (in degrees, such that $0 \le \theta \le 180$) is ____.

[EC, 2016 : 1 Mark, Set-2]

Solution:

$$I = \hat{x}15\cos\omega t + \hat{y}5\sin\omega t$$

$$|I| = \sqrt{(15\cos\omega t)^2 + (5\sin\omega t)^2}$$

$$= \sqrt{225\cos^2\omega t + 25\sin^2\omega t} = \sqrt{25 + 200\cos^2\omega t}$$

$|I|$ is minimum when $\cos^2\omega t = 0$

or $\theta = \omega t = 90°$

2.14.9 Inner Product (Dot Product)

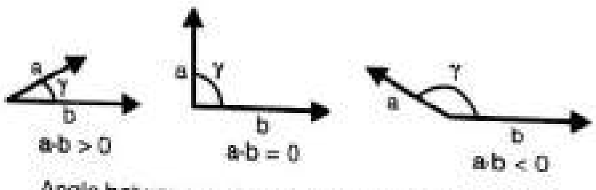
We shall now define a multiplication of two vectors that gives a scalar as the product and is suggested by various applications.

Definition. Inner Product (Dot Product) of Vectors

The inner product or dot product **a**. **b** (read "**a** dot **b**") of two vectors **a** and **b** is the product of their lengths times the cosine of their angle, see Fig. below.

1. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\gamma$

   The angle γ, $0 \le \gamma \le \pi$, between a and b is measured when the vectors have their initial points coinciding, as in Fig. below.



Angle between vectors and value of inner product

In components, $a = [a_1, a_2, a_3]$, $b = [b_1, b_2, b_3]$, and

2. $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

   can be derived from (i).

   Since the cosine in (i) may be positive, zero, or negative, so may be the inner product. The case that the inner product is zero is of great practical interest and suggests the following concept.

A vector a is called orthogonal to a vector b if a.b = 0. Then b is also orthogonal to a and we call these vectors orthogonal vectors. Clearly, the zero vector is orthogonal to every vector.

For nonzero vectors we have $\dot{a}.\dot{b} = 0$ if and only if $\cos\gamma = 0$; thus $\gamma = \pi/2(90°)$. This proves the following important theorem.

Theorem: 1 (Orthogonality)

The inner product of two nonzero vectors is zero if and only if these vectors are perpendicular.

Length and Angle in Terms of Inner Product: Equation (i) above with $\dot{b} = \dot{a}$ gives $a.a = |a|^2$.

3. $|a| = \sqrt{a \cdot a}$

From (i) and (iii) we obtain for the angle γ between two nonzero vectors

4. $\cos\gamma = \dfrac{a \cdot b}{|a||b|} = \dfrac{a \cdot b}{\sqrt{a \cdot a}\sqrt{b \cdot b}}$

ILLUSTRATIVE EXAMPLES

Example:

Find the inner product and the lengths of $\dot{a} = [1, 2, 0]$ and $\dot{b} = [3, -2, 1]$ as well as the angle between these vectors.

Solution:

$$a.b = 1.3 + 3.(-2) + 0.1 = -1$$

$$|a| = \sqrt{a \cdot a} = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

$$|b| = \sqrt{b \cdot b} = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

$$\gamma = \text{arc } \cos\frac{a \cdot b}{|a||b|} = \text{arc}\cos(-0.11952)$$

$$= 1.69061 = 96.865°$$

The given vectors make an obtuse angle between them solves and notice that the inner product has come out negative because of this.

General Properties of Inner Products: From the definition we see that the inner product has the following properties. For any vectors a, b, c and scalars q_1, q_2.

(a) $[q_1 a + q_2 b].\dot{c} = q_1 a.\dot{c} + q_2 b.\dot{c}$ (Linearity)

(b) a.b = b.a (Symmetry)

(c) $a . a \geqq 0$ (Positive-definiteness)

(d) a . a = 0 if and only if a = 0 (Positive-definiteness)

Hence dot multiplication is commutative and is distributive with respect to vector addition; in fact, from above (a) with $q_1 = 1$ and $q_2 = 1$ we have

5. $(\dot{a} + \dot{b}).\dot{c} = \dot{a}.\dot{c} + \dot{b}.\dot{c}$ (Distributivity)

Furthermore, from $a.b = |\dot{a}||\dot{b}| \cos\gamma$ and $|\cos\gamma| \leq 1$, So

6. $|a.b| \leq |a|\,|b|$ (Schwarz inequality)

7. $|a + b| \leq |a| + |b|$ (Triangle inequality)

A simple direct calculation with inner products shows that

8. $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$ (Parallelogram equality)
   Equations (6) – (8) play a basic role in so-called Hilbert spaces (abstract inner product spaces), which form the basis of quantum mechanics.

Derivation of $a.b = a_1b_1 + a_2b_2 + a_3b_3$ from $a.b = |a|\,|b|\cos\gamma$

Let $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$.

Since i, j, and k are unit vectors, we have from (3) $i.i = |i|^2 = 1$, $j.j = |j|^2 = 1$ and $k.k = |k|^2 = 1$.

Since i, j, k are or orthogonal to each other (The coordinate axes being perpendicular to each other), we get from theorem, $i.j = 0$, $j.k = 0$, $k.i = 0$.

Now,
$$a.b = (a_1i + a_2j + a_3k).(b_1i + b_2j + b_3k)$$

using distributive property, we first have a sum of nine inner products.
$$a.b = a_1b_1i.i + a_1b_2i.j + ... + a_3b_3k.k.$$

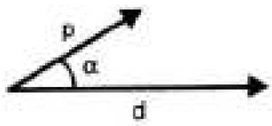
Since six of these products are zero, we obtain $a.b = a_1b_1 + a_2b_2 + a_3b_3$

Applications of Inner Products: Typical applications of inner products are shown in the following examples.

ILLUSTRATIVE EXAMPLES

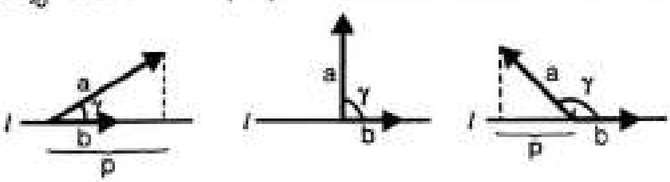
Example:
Work done by a force as inner product.

Solution:
Consider a body on which a constant force **p** acts. Let the body be given a displacement **d**. Then the work done by **p** in the displacement is defined as
$$W = |p||d|\cos\alpha = p.d$$
that is, magnitude $|p|$ of the force times length $|d|$ of the displacement times the cosine of the angle a between p and d. If a < 90°, as in Fig. below then W > 0. If p and d are orthogonal, then the work done is zero. If a > 90°, then W < 0, which means that in the displacement one has to do work against the force.



Work done by a force

Vector Projection: $Proj_b^a$ is the vector projection of a on another vector b.



$$p = Proj_b^a$$
= (Scalar component of a in direction of b) × (a unit vector in direction of b)

$$Proj_b^a = (|a|\cos\gamma)\left(\frac{b}{|b|}\right)$$
$$= \left(\frac{a\cdot b}{|b|}\right)\left(\frac{b}{|b|}\right) = \left(\frac{a\cdot b}{bb}\right)b$$

Typical application of projection is finding component of n force in a given direction as is often required in mechanics.

ILLUSTRATIVE EXAMPLES

Example:

Vector projection of a on another vector b.

Find the vector projection of a vector a = 2i − 3j or b = 3i + 4j.

Solution:

$$\text{Proj}_b{}^a = \left(\frac{a.b}{b.b}\right)b = \left(\frac{2.3-3.4}{3.3+4.4}\right)(3i+4j) = \frac{-6}{25}(3i+4j) = \frac{-18}{25}i - \frac{24}{25}j$$

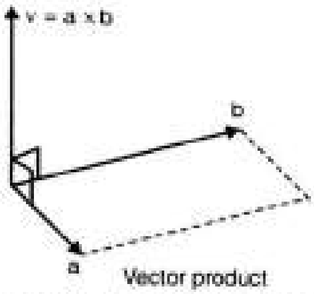
2.14.10 Vector Product (Cross Product)

The dot product is a scalar. We shall see that some applications, for instance, in connection with rotations, require a product of two vector which is again a vector. This is called vector product of two vectors or the cross product.

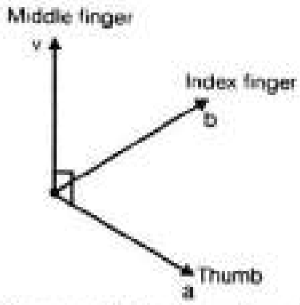
Definition. Vector product (Cross product)

The **vector product (cross product) a × b** of two vectors **a** = $[a_1, a_2, a_3]$ and **b** = $[b_1, b_2, b_3]$ is a vector.

$v = a \times b = |a|\,|b| \sin\gamma\, \hat{n}$ such that $\dot{a}$, $\dot{b}$ and $\hat{n}$ from a right handed system, with $\hat{n}$ being a unit normal vector perpendicular to plane of a and b.



Vector product



If a and b have the same or opposite direction or if one of these vectors is the zero vector, then **v** = **a** × **b** = **0**. In any other case, v = a × b has the length.

1. $|\dot{v}| = |\dot{a}|\,|\dot{b}| \sin\gamma$

   This is the area of the parallelogram in Figure above with $\dot{a}$ and $\dot{b}$ as adjacent sides. (γ is the angle between a and b). The direction of **v** = **a** ×**b** is perpendicular to both a and b and such that a, b, v, in this order, form a right-handed triple as shown in figure above.

   In components, v = $[v_1, v_2, v_3]$ = **a × b** is

2. $v_1 = a_2b_3 - a_3b_2, \quad v_2 = a_3b_1 - a_1b_3, \quad v_3 = a_1b_2 - a_2b_1$

   i.e. If a is in direction of (right hand) thumb, b is in direction of index figure, then v = a × b will be a vector in direction of the middle figure.

   In terms of determinants:

$$v_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad v_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \qquad v_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Hence $v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k$ is the expansion of the symbolical third-order determinant

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

by the first row. (We call it "symbolical" because the first row consists of vectors rather than numbers.)

2.14.10.1 Finding a Unit Vector Perpendicular to two Given Vectors a and b

A unit vector perpendicular to two given vectors a and b is given by

$$n = \frac{a \times b}{|a||b|\sin\gamma} = \frac{a \times b}{|a \times b|}$$

ILLUSTRATIVE EXAMPLES

Example: 1

With respect to a right-handed Cartesian coordinate system, let a = [4, 0, −1] and b = [−2, 1, 3].

Solution:

$$a \times b = \begin{vmatrix} i & j & k \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{vmatrix} = i - 10j + 4k = [1, -10, 4]$$

Example: 2

Find a unit vector perpendicular to both a = 3i + j + 2k and b = 2i − 2j + 4k.

Solution:

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8i - 8j - 8k$$

A unit vector perpendicular to both a and b is

$$n = \frac{a \times b}{|a \times b|}$$

$$= \frac{8i - 8j - 8k}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(i - j - k)$$

There are z unit vectors perpendicular to both a and b. They are $\pm n = \pm\frac{1}{\sqrt{3}}(i - j - k)$

Example: 3

The vectors from origin to the points A and B are $\vec{a} = \hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$ respectively. Find the area of

(a) the triangle OAB

(b) the parallelogram formed by $\overline{OA}$ and $\overline{OB}$ as adjacent sides.

Solution:

Given $\overline{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\overline{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$.

$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (12 - 2)\hat{i} - (-6 - 4)\hat{j} + (3 + 12)\hat{k} = 10\hat{i} + 10\hat{j} + 15\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}.$$

(a) area of $\Delta OAB = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\cdot 5\sqrt{17}$ sq. units $= \frac{5}{2}\sqrt{17}$ sq. units.

(b) Area of parallelogram formed by $\overline{OA}$ and $\overline{OB}$ as adjacent sides

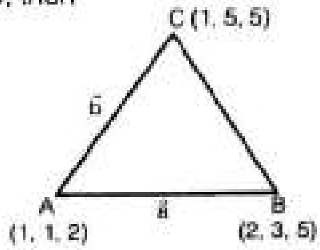
$$= |\vec{a}\times\vec{b}| = 5\sqrt{17} \text{ sq. units.}$$

Example: 4

Using vectors, find the area of the triangle with vertices A(1, 1, 2), B (2, 3, 5) and C(1, 5, 5)

Solution:

Let the vectors $\vec{a}$ and $\vec{b}$ represents the sides AB and AC of ΔABC, then



$$\vec{a} = \overline{AB} = \text{P.V. of B} - \text{P.V. of A}$$
$$= (2\hat{i}+3\hat{j}+5\hat{k})-(\hat{i}+\hat{j}+2\hat{k})$$
$$= \hat{i}+2\hat{j}+3\hat{k}$$

and $\vec{b} = \overline{AC} = \text{P.V. of C} - \text{P.V. of A}$

$$= (\hat{i}+5\hat{j}+5\hat{k})-(\hat{i}+\hat{j}+2\hat{k}) = 4\hat{j}+3\hat{k}$$

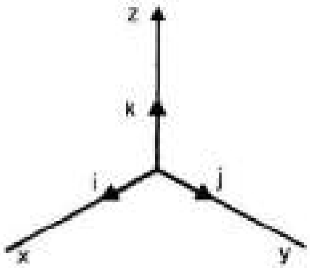
No $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 0 & 4 & 3\end{vmatrix} = (6-12)\hat{i}-(3-0)\hat{j}+(4-0)\hat{k}$

$$= -6\hat{i}-3\hat{j}+4\hat{k}$$

$\Rightarrow$ $|\vec{a}\times\vec{b}| = \sqrt{(-6)^2+(-3)^2+4^2} = \sqrt{61}$

$\therefore$ The area of $\Delta ABC = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\sqrt{61}$

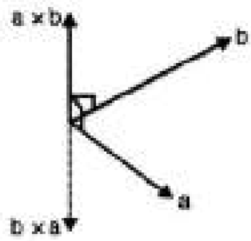
2.14.10.2 Vector Products of the Standard Basis Vectors



Since i, j, k are orthogonal (mutually perpendicular) unit vectors, the definition of vector product gives some useful formulas for simplifying vector products: in right-handed coordinates these are

i × j = k	**j × k = i,**	**k × i = j**
j × i = – k	**k × j = –i,**	**i × k = – j.**

2.14.10.3 General Properties of Vector Products



Vector Product has the property that for every scalar *l*,

$$(l\mathbf{a})\times\mathbf{b} = l(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(l\mathbf{b}).$$

It is distributive with respect to vector addition, that is,

$$\dot{a}\times(\dot{b}+\dot{c}) = (\dot{a}\times\dot{b})+(\dot{a}\times\dot{c}),$$
$$(\dot{a}+\dot{b})\times\dot{c} = (\dot{a}\times\dot{c})+(\dot{b}\times\dot{c}).$$

It is **not commutative** but **anticommutative**, that is,

$$b \times a = -(a \times b)$$

It is not associative, that is,

$$a \times (b \times c) \neq (a \times b) \times c \qquad \text{(in general)}$$

so that the parentheses cannot be omitted.

2.14.11 Scalar Triple Product

The scalar triple product or mixed triple product of three vectors

$a = [a_1, a_2, a_3]$, $b = [b_1, b_2, b_3]$, $c = [c_1, c_2, c_3]$

is denoted by (a b c) and is defined by (a b c) and is defined by **(a b c) = a.(b × c)**

We can write this as a third-order determinant. For this we set $b \times c = v = [v_1, v_2, v_3]$. Then from the dot product in components we obtain

$$a.(b \times c) = a.v = a_1v_1 + a_2v_2 + a_3v_3$$

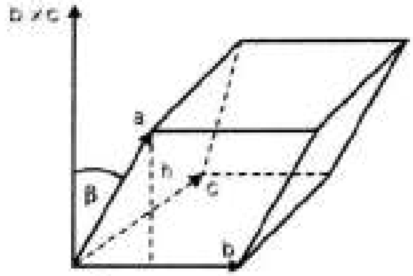
$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\left(-\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}\right) + a_3\begin{vmatrix} b_1 & b_2 \\ c_2 & c_2 \end{vmatrix}$$

The expression on the right is the expansion of a third-order determinant by its first row. Thus

$$(a \ b \ c) = a\cdot(b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric Interpretation of Scalar Triple Products

The absolute value of the scalar triple product is the volume of the parallelepiped with a, b, c as edge vectors (Figure below, $|a.(b \times c)| = |a||b \times c|\cos\beta$ where $|a|\ |\cos\beta|$ is the height h and, by (1), the base, the parallelogram with sides b and c, has area $|b \times c|$. Naturally, if vectors a, b and c are coplanar, then this volume is zero. $a(b \times c) = 0$, if a, b and c are coplanar.



Geometrical interpretation
of a scalar triple product

we also have for any scalar k,

$$\textbf{(ka b c) = k(a b c)}$$

because the multiplication of a row of a determinant by k multiples the value of the determinant by k. Furthermore, we prove that

$$\textbf{a.(b × c) = (a × b).c}$$

Proof: LHS of above $= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\text{RHS of above} = (a \times b) \cdot c = c \cdot (a \times b) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

By properties of determinants it can be seen that the LHS and RHS determinants are indeed both equal.

So, $\quad a \,.\, (b \times c) = (a \times b) \,.\, c$

In fact $\quad a(b \times c) = b.(c \times a) = c.\,(a \times b)$

i.e. the value of triple product depends upon the cycle order of the vectors, but is independent of the position of dot and cross. However if the order is non-cycle, then value changes.

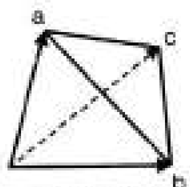
i.e. $\quad a.(b \times c) \neq b.(a \times c)$

ILLUSTRATIVE EXAMPLES

Example:

A tetrahedron is determined by three edge vectors a, b, c as indicated in Fig. below.

Find its volume if with respect to right-handed Cartesian coordinates, a = [2, 0, 3], b = [0, 6, 2], c = [3, 3, 0].



Tetrahedron

Solution:

The volume V of the parallelopiped with these vectors as edge vectors is the absolute value of the scalar triple product.

$$(a \quad b \quad c) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 6 & 2 \\ 3 & 3 & 0 \end{vmatrix} = 2\begin{vmatrix} 6 & 2 \\ 3 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 6 \\ 3 & 3 \end{vmatrix} = -12 - 54 = -66$$

That is, V = 66. The minus sign indicates that a, b, c, in this order, form a left-handed triple. The volume of the tetrahedron is $\frac{1}{6}$ of that of the parallelepiped, hence 11.

Testing Linear Independence of 3 Vectors using Scalar Triple Product:

Linear independence of three vectors can be tested by scalar triple products, as follows. We call a given set of vectors $a_{(1)}, \dots, a_{(m)}$ linearly independent if the only scalar $c_1, \dots, c_m$ for which the vector equation.

$$c_1 a_{(1)} + c_2 a_{(2)} + \dots + c_m a_{(m)} = 0$$

is satisfied are $c_1 = 0$, $c_2 = 0$, ..., $c_m = 0$. otherwise, that is, if that equation also holds for an m-tuple of scalars not all zero, we call that set of vectors linearly dependent.

Now three vectors, if we let their initial point coincide, form a linearly independent set if and only if they do not lie in the same plane (or on the same line). i.e. These vectors are linearly independent, if and only if they are not co-planar. The interpretation of a scalar triple product as a volume thus gives the following criterion.

Theorem: 1 (Linear Independence of Three Vectors)

Three vectors form a linearly independent set if and only if their scalar triple product is not zero. The scalar triple product is the most important "repeated product." Other repeated product exist, but are used only occasionally.

2.14.12 Vector Triple Product

If a, b and c are three vectors then the vector triple product is written as $a \times (b \times c)$

It can be proved that $a \times (b \times c) = (a.c)b - (a.b)c$

ILLUSTRATIVE EXAMPLES

Example:

Let $a = i + j - k$, $b = i - j + k$; $c = i - j - k$

Find the vector $a \times (b \times c)$

Solution:

Since,
$$a \times (b \times c) = (a.c)b - (a.b)c$$
$$a.c = 1 - 1 + 1 = 1$$
$$a.b = 1 - 1 - 1 = -1$$

So,
$$a \times (b \times c) = 1 \,.\, b - (-1)\,.\, c = b + c$$
$$= (i - j + k) + (i - j - k) = 2i - 2j$$

This is the end of vector algebra (in 3-space and in the plane). Vector calculus (i.e. differentiation of vectors) begins in the next section.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.148 If P, Q and R are three points having coordinates (3, –2, –1), (1, 3, 4), (2, 1, –2) in XYZ space, then the distance from point P to plane OQR (O being the origin of the coordinate system) is given by

(a) 3
(b) 5
(c) 7
(d) 9

[CE, GATE-2003, 1 mark]

Solution: (a)

Solution by Coordinate Geometry:

This problem can be done through coordinate geometry formula or through vectors.

Given,
P (3, –2, –1)
Q (1, 3, 4)
R (2, 1, –2)
O (0, 0, 0)

Equation of plane OQR is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 0 & y - 0 & z - 0 \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

i.e.
$$2x - 2y + z = 0$$

Now ⊥ distance of (x_1, y_1, z_1) from $ax + by + cz + d = 0$ is given by

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Therefore, ⊥ distance of (3, –2, –1) from plane

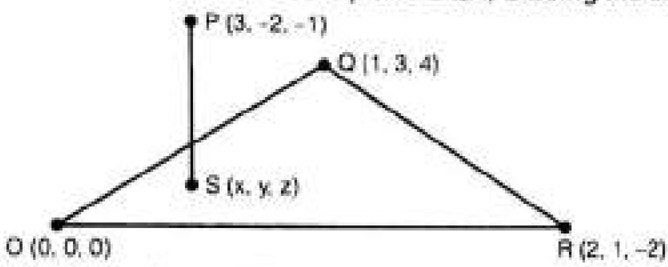
$$2x - 2y + z = 0 \text{ is given by}$$

$$\left|\frac{2 \times 3 - 2 \times (-2) + (-1)}{\sqrt{2^2 + (-2)^2 + 1^2}}\right| = 3$$

Solution by Vectors:

Given, P (3, –2, –1)
Q (1, 3, 4)
R (2, 1, –2)
O (0, 0, 0)

We wish to find the distance of point P to the plane OQR, O being the origin. See figure below.



Let us drop a perpendicular from pt P(3, –2, –1) to plane OQR. Let it meet at pt S(x, y, z). Now we wish to find length of PS. For this we must first find x, y and z values. Findly, S is on the plane OQR. OS is coplanar with OQ OR. This gives us $\overline{OS}.(\overline{OQ} \times \overline{OR}) = 0$.

Since the position vector of OS is xi + yi + zk, the position vector of OQ is i + 3i + 4k and position vector of OR is 2i + i – 2k.

$$\therefore \qquad \overline{OS}\,.\,(\overline{OQ} \times \overline{OR}) = \begin{vmatrix} x & y & z \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \qquad -10x + 10y - 5z = 0 \qquad \text{... (i)}$$

Also PS is normal to plane OQR and so normal to $\overline{OQ}$ and $\overline{OR}$.

A vector normal to $\overline{OQ}$ and $\overline{OR}$ is $\overline{OQ} \times \overline{OR}$.

$$\overline{OQ} \times \overline{OR} = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix} = -10i + 10j - 5k$$

Now the vector $\overline{PS}$ is given by (3 – x)i + (–2 – y)j + (–1 – z)k. Now $\overline{PS}$ must have same direction ratios as a vector normal to $\overline{OQ}$ and $\overline{OR}$ since $\overline{PS}$ itself is normal to $\overline{OQ}$ and $\overline{OR}$.

So, $3 - x : -2 - y : -1 - z = -10 : 10 : -5$

This results in 2 equations

$$\frac{3-x}{-2-y} = \frac{-10}{10}$$

$$\Rightarrow \qquad x + y = 1 \qquad \text{... (ii)}$$

and $$\frac{-2-y}{-1-z} = \frac{10}{-5}$$

$\Rightarrow$ $$y + 2z = -4 \quad \text{...(iii)}$$

Now, we have 3 equations in 3 unknowns which can be solved.

Substituting x and z in terms of y in to equation (i) we get.

$$-10(1-y) + 10y - 5\left(\frac{-4-y}{2}\right) = 0$$

$\Rightarrow$ $$y = 0$$

$\therefore$ from equation (ii), $x = 1$ and from equation (iii), $z = -2$

So, the point S is given by $S(1, 0, -2)$

Now, since P is given by $P(3, -2, -1)$

$$\vec{PS} = (3-1)i + (-2-0)j + (-1+2)k$$
$$= 2i - 2j + k$$
$$\left|\vec{PS}\right| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$

Q.149 The inner (dot) product of two non zero vectors $\vec{P}$ and $\vec{Q}$ is zero. The angle (degrees) between the two vectors is

(a) 0
(b) 30
(c) 90
(d) 120

[CE, GATE-2008, 2 marks]

Solution: (c)

$$\vec{P} \cdot \vec{Q} = 0$$
$$\vec{P} \cdot \vec{Q} = |P|\,|Q| \cos\theta$$

if $$\vec{P} \cdot \vec{Q} = 0$$

$\Rightarrow$ $$|P|\,|Q| \cos\theta = 0$$

Since, P and Q are non-zero vectors

$\Rightarrow$ $$\cos\theta = 0$$

$\Rightarrow$ $$\theta = 90^\circ$$

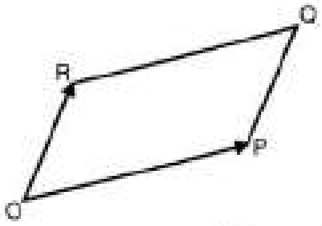
Q.150 If $\vec{a}$ and $\vec{b}$ are two arbitrary vectors with magnitudes a and b, respectively, $\left|\vec{a} \times \vec{b}\right|^2$ will be equal to

(a) $a^2b^2 - \left(\vec{a} \cdot \vec{b}\right)^2$
(b) $ab - \vec{a} \cdot \vec{b}$
(c) $a^2b^2 + \left(\vec{a} \cdot \vec{b}\right)^2$
(d) $ab + \vec{a} \cdot \vec{b}$

[CE, GATE-2011, 2 marks]

Answer: (a)

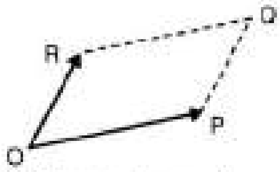
Q.151 For the parallelogram OPQR shown in the sketch, $\overline{OP} = a\hat{i} + b\hat{j}$ and $\overline{OR} = c\hat{i} + d\hat{j}$. The area of the parallelogram is



(a) ad − bc
(b) ac + bd
(c) ad + bc
(d) ab − cd

[CE, GATE-2012, 2 marks]

Solution: (a)



The area of parallelogram OPQR in figure shown above, is the magnitude of the vector product

$$= \left|\overline{OP} \times \overline{OR}\right|$$

$$\overline{OP} = a\hat{i} + b\hat{j}$$

$$\overline{OR} = e\hat{i} + d\hat{j}$$

$$\overline{OP} \times \overline{OR} = \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (ad - bc)\hat{k}$$

$$\left|\overline{OP} \times \overline{OR}\right| = \sqrt{0^2 + 0^2 + (ad - bc)^2} = ad - bc$$

Q.152 The angle between two unit-magnitude coplanar vectors P(0.866, 0.500, 0) and Q (0.259, 0.966, 0) will be

(a) 0°
(b) 30°
(c) 45°
(d) 60°

[ME, GATE-2004, 1 mark]

Solution: (c)

$$\vec{P} = 0.866\hat{i} + 0.500\hat{j} + 0\hat{k}$$

$$\vec{Q} = 0.259\hat{i} + 0.966\hat{j} + 0\hat{k}$$

$\therefore$ $$\vec{P}.\vec{Q} = |\vec{P}|\,|\vec{Q}|\cos\theta$$

Here, $|P| = |Q| = 1$ (unit magnitude)

So, $(0.866\hat{i} + 0.5\hat{j} + 0\hat{k}).(0.259\hat{i} + 0.966\,\hat{j} + 0\hat{k})$

$$= \sqrt{(0.866)^2 + (0.5)^2} \times \sqrt{(0.259)^2 + (0.966)^2}\,.\cos\theta$$

$\therefore$ $$\cos\theta = \frac{0.866 \times 0.259 + 0.5 \times 0.966}{\sqrt{1} \times \sqrt{1}} = 0.707$$

$\therefore$ $$\theta = 45°$$

Q.153 The area of a triangle formed by the tips of vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is

(a) $\frac{1}{2}(\bar{a}-\bar{b}).(\bar{a}-\bar{c})$

(b) $\frac{1}{2}|(\bar{a}-\bar{b})\times(\bar{a}-\bar{c})|$

(c) $\frac{1}{2}|\bar{a}\times\bar{b}\times\bar{c}|$

(d) $\frac{1}{2}(\bar{a}\times\bar{b}).\bar{c}$

[ME, GATE-2007, 2 marks]

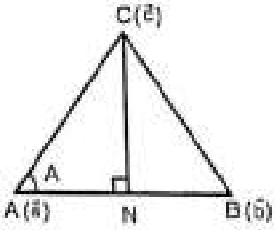
Solution: (b)

Fom C, draw CN ⊥ AB. From right-angled ΔCAN,

$$\sin A = \frac{|CN|}{|AC|} \Rightarrow |CN| = |AC| \sin A.$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|AB|\times|CN|$$

$$= \frac{1}{2}|AB|\cdot|AC|\sin A = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$



From above figure, $\overrightarrow{AB} = \vec{b}-\vec{a}$ and $\overrightarrow{AC} = \vec{c}-\vec{a}$.

So, Area of $\Delta ABC = \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})| = \frac{1}{2}|(\bar{a}-\bar{b})\times(\bar{a}-\bar{c})|$

Choice (b) is correct.

Q.154 Let x and y be two vectors in a 3 dimensional space and <x, y> denote their dot product. Then the determinant det$\begin{bmatrix} <x,x> & <x,y> \\ <y,x> & <y,y> \end{bmatrix}$.

(a) is zero when x and y are linearly independent

(b) is positive when x and y are linearly independent

(c) is non-zero for all non-zero x and y

(d) is zero only when either x or y is zero

[EE, GATE-2007, 2 marks]

Solution: (b)

Let $$D = \begin{vmatrix} x.x & x.y \\ y.x & y.y \end{vmatrix}$$

Let $$x = x_1\,\hat{i} + x_2\,\hat{j}$$

$$y = y_1\,\hat{i} + y_2\,\hat{j}$$

$$x\,.\,x = x_1^2 + x_2^2$$

$$y\,.\,y = y_1^2 + y_2^2$$

$$x\,.\,y = x_1\,x_2 + y_1\,y_2$$

∴ $$D = \begin{vmatrix} x_1^2 + x_2^2 & x_1 x_2 + y_1 y_2 \\ x_1 x_2 + y_1 y_2 & y_1^2 + y_2^2 \end{vmatrix}$$

$$= (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1 x_2 + y_1 y_2)^2$$
$$= x_2^2 y_1^2 + x_1^2 y_2^2 - 2x_1 y_1 x_2 y_2$$
$$= (x_2 y_1 - x_1 y_2)^2$$

Now, $D = 0$

$$x_2 y_1 - x_1 y_2 = 0$$

$\Rightarrow$ $$\frac{x_1}{x_2} = \frac{y_1}{y_2}$$

$\Rightarrow$ Vector $x_1 i + x_2 j$ and $y_1 i + y_2 j$ are linearly dependent.

$\therefore$ Linear dependence $\Rightarrow D = 0$

So, Linear independence $\Rightarrow D \neq 0$

i.e. is negetive or positive.

However, [notice that here since $D = (x_2 y_1 - x_1 y_2)^2$, it cannot be negative].

So, Linear independence $\Rightarrow$ D is positive.

Q.155 The two vectors [1, 1, 1] and [1, a, a^2], where $a = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, are

(a) orthonormal
(b) orthogonal
(c) parallel
(d) collinear

[EE, GATE-2011, 2 marks]

Solution: (b)

Given [1, 1, 1] and [1, a, a^2]

hence $$a = \omega = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$
$$a^2 = \omega^2$$

So the vectors we

$$u = [1, 1, 1]$$

and $$v = [1, \omega, \omega^2]$$

Now $$u.v = 1.1 + 1.\omega + 1.\omega^2$$
$$= 1 + \omega + \omega^2 = 0$$

So u & v are orthogonal.

4. The vector that is NOT perpendicular to the vectors $(i + j + k)$ and $(i + 2j + 3k)$ is_____.

(a) $(i - 2j + k)$
(b) $(-i + 2j - k)$
(c) $(0i + 0j + 0k)$
(d) $(4i + 3j + 5k)$

[IN, 2016 : 1 Mark]

Solution: (d)

We know that if $\vec{a}$ and $\vec{b}$ are perpendicular

then $\vec{a} \cdot \vec{b} = 0$

options (a), (b), (c) are perpendicular.

options (d) is not perpendicular.

2.14.13 Vector and Scalar Functions and Fields. Derivatives

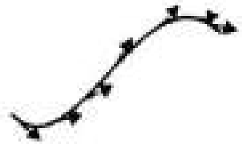
This is the beginning of vector calculus, which involves two kinds of functions, vector functions, whose values are vectors.

$$v = v(P) = [v_1(P), v_2(P), v_3(P)]$$

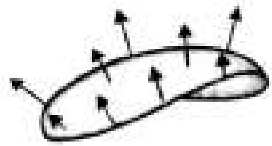
depending on the points P in space, and scalar functions, whose values are scalars

$$f = f(P)$$

depending on P. In applications, the domain of definition for such a function is a region of space or a surface in space or a curve in space. We say that a vector function defines a vector field in that region (or on that surface or curve). Examples are shown in figures. Similarly, a scalar function defines a scalar field in a region or on a surface or a curve. Examples, are the temperature field in a body (scalar function) and the pressure field of the air in the earth's atmosphere. Vector (vector function) and scalar functions may also depends on time t or on further parameters.



Field of tangent vectors of a curve



Field of normal vectors of a surface

Comment on Notation. If we introduce cartesion coordinates x, y, z, then instead of v(P) and f(P) we can also write

$$v(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$$

and f(x, y, z), but we keep in mind that a vector or scalar field that has a geometrical or physical meaning should depend only on the points P where it is defined but not on the particular choice of Cartesian coordinates.

ILLUSTRATIVE EXAMPLES

Example: 1

Scalar function (Euclidean distance in space).

Solution:

The distance f(P) of any point P from a fixed point P_0 in space is a scalar function whose domain of definition is the whole space. f(P) defines a scalar field in space. If we introduce a Cartesian coordinate system and P_0 has the coordinates x_0, y_0, z_0 then f is given by the well-known formula

$$f(P) = f(x, y, z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

where x, y, z are the coordinates of P. If we replace the given Cartesian coordinate system by another such system, then the values of the coordinates of P and P_0 will in general change, but f(p) will have the same value as before. Hence f(P) is a scalar function. The direction cosines of the line through P and P_0 are not scalars because their values will depend on the choice of the coordinate system.

Example: 2

Vector field (Velocity field).

Solution:

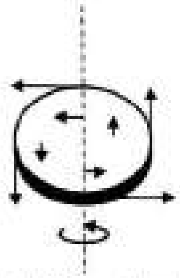
At any instant the velocity vectors v(P) of a rotating body B constitute a vector field, the so-called velocity field of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotation, then

$$v(x, y, z) = w \times r = w \times [x, y, z] = w \times (xi + yj + zk)$$

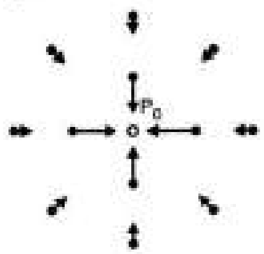
where x, y, z are the coordinates of any point P of B at the instant under consideration. If the coordinates are such that the z-axis is the ais of rotation and w points in the positive z-direction, then w = ωk and

$$v = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega(-yi + xj) = \omega[-y, x, 0]$$

An example of a rotating body and the corresponding velocity field are shown in Figure below. Also shown is another example of vector field, the gravitational field.



Velocity field of a rotating body



Gravitational field

Vector Calculus: We show next that the basic concepts of calculus, such as convergence, continuity, and differentiability, can be defined for vector functions in a simple and natural way. Most important here is the derivative.

Convergence: An infinite sequence of vectors $a_{(n)}$, n = 1, 2, ..., is said to **converge** if there is a vector a such that

$$\lim_{n \to \infty} |\vec{a}_{(n)} - \vec{a}| = 0$$

a is called the limit vector of that sequence, and we write

$$\lim_{n \to \infty} \vec{a}_{(n)} = \vec{a}$$

Cartesian coordinates being given, this sequence of vectors converges to a if and only if the three sequences of components of the vectors converge to the corresponding components of a.

Similarly, a vector function v(t) of a real variable t is said to have the limit l as t approaches t_0, if v(t) is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t \to t_0} |v(t) - l| = 0$$

Then we write, $$\lim_{t \to t_0} v(t) = l$$

Continuity: A vector function v(t) is said to be continuous at $t = t_0$ if it is defined in some neighborhood of t_0 and

$$\lim_{t \to t_0} v(t) = v(t_0)$$

If we introduce a Cartesian coordinate system, we may write

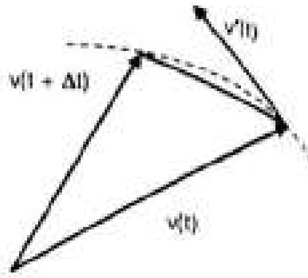
$$v(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)i + v_2(t)j + v_3(t)k.$$

Then v(t) is continuous at t_0 if and only if its three components are continuous at t_0. We now state the most important of these definitions.

2.14.13.1 Derivative of a Vector Function

A vector function v(t) is said to be differentiable at a point t if the following limit exists:

$$v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$



Derivative of a vector function

exists. The vector v'(t) is called the derivative of v(t). See Figure above (The curve in this figure is the locus of the heads of the arrows representing v for values of the independent variable in some interval containing t and t + Δt).

In terms of components with respect to a given Cartesian coordinate system v(t) is differentiable at a point t if and only if its three components are differentiable at t, and then the derivative v'(t) is obtained by differentiating each component separately.

$$v'(t) = [v_1'(t),\ v_2'(t),\ v_3'(t)]$$

It follows that the familiar rules of differentiation yield corresponding rules for differentiating vector functions, for example,

$$(cv)' = cv' \qquad \text{(c constant)}$$
$$(u + v)' = u' + v' \text{ and in particular.}$$
$$(u.v)' = u'.v + u.v'$$
$$(u \times v)' = u' \times v + u \times v'$$
$$(u\ \ v\ \ w)' = (u'\ \ v\ \ w) + (u\ \ v'\ \ w) + (u\ \ v\ \ w')$$

The order of the vectors must be carefully observed because cross multiplication is not commutative.

ILLUSTRATIVE EXAMPLES

Example:

Derivative of a vector function of constant length.

Solution:

Let v(t) b a vector function whose length is constant, say, $|v(t)| = c$. Then $|v|^2 = v.v = c^2$, and $(v.v)' = v'v + vv' = 2v.v' = 0$, by differentiation. The yields the following result. The derivative of a vector function v(t) of constant length is either the zero vector or is perpendicular to v(t)

2.14.13.2 Partial Derivatives of a Vector Function

From our present discussion we see that partial differentiation of vector functions depending on two or more variables can be introduced as follows. Suppose that the components of a vector function

$$v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k$$

are differentiable functions of n variables $t_1, \ldots, t_n$. Then the partial derivative of v with respect to t_l is denoted by $\partial v/\partial t_l$ and is defined as the vector function

$$\frac{\partial v}{\partial t_l} = \frac{\partial v_1}{\partial t_l} i + \frac{\partial v_2}{\partial t_l} j + \frac{\partial v_3}{\partial t_l} k$$

Similarly,

$$\frac{\partial^2 v}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m} i + \frac{\partial^2 v_2}{\partial t_l \partial t_m} j + \frac{\partial^2 v_3}{\partial t_l \partial t_m} k \text{ and so on.}$$

ILLUSTRATIVE EXAMPLES

Example:

Let $r(t_1, t_2) = a\cos t_1\, i + a\sin t_1 j + t_2 k.$

Solution:

Then
$$\frac{\partial r}{\partial t_1} = -a\sin t_1\, i + a\cos t_1 j$$
$$\frac{\partial r}{\partial t_2} = k$$

Various physical and geometrical applications of derivatives of vector functions will be discussed in the next sections.

2.14.14 Gradient of a Scalar Field

We shall see that some of the vector fields in applications-(not all of them) can be obtained from scalar fields. This is a considerable advantage because scalar fields can be handled more easily. The relation between the two types of fields is accomplished by the "gradient." Hence the gradient is of great practical importance.

Definition of Gradient: The gradient grad f of a given scalar function f(x, y, z) is the vector function defined by

1. $\text{grad } f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

   Here we must assume that f is differentiable. It has become popular, particularly with physicists and engineers, to introduce the differential operator.

2. $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

   (read nabla or del) and to write

   $\text{grad } f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

   For instance, if $f(x, y, z) = 2x + yz - 3y^2$, then $\text{grad } f = \nabla f = 2i + (z - 6y)j + yk$.

   We show later that grad f is a vector; that is, although it is defined in terms of components, it has a length and direction that is independent of the particular choice of Cartesian coordinates. But first we explore how the gradient is related to the rate of change of f in various directions. In the directions of the three coordinate axes, this rate is given by the partial derivatives, as we know from calculus. The idea of extending this to arbitrary directions seems natural and leads to the concept of directional derivative.

2.14.15 Directional Derivative

The rate of change of f at any point P in any fixed direction given by a vector b is defined as in calculus. We denote it by $D_b f$ or df/ds, call it the directional derivative of f at P in the direction of b, and define it by figure.

3. $D_b f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$ (s = distance between P and Q)

   where Q is a variable point on the ray C in the direction of b as in Fig. below.

   The next idea is to use Cartesian xyz-coordinates and for b a unit vector. Then the ray C is given by

4. $r(s) = x(s)i + y(s)j + z(s)k = p_0 + sb \qquad (s \geq 0, |b| = 1)$

(p_0 the position vector of P). Equation (3) now shows that $D_b f = df/ds$ is the derivative of the function f(x(s), y(s), z(s)) with respect to the arc length s of C. Hence, assuming that f has continuous partial derivatives and applying the chain rule. We obtain

Directional derivative

5. $D_b f = \frac{df}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'$

where primes denote derivatives with respect to s (which are taken at s = 0). But here, $r' = x'i + y'j + z'k = b$ by (4). Hence (5) is simply the inner product of b and grad f [see (2), Sec. 8.2].

6. $D_b f = \frac{df}{ds} = b.\text{grad } f \qquad (|b| = 1)$

Attention! In general, if the direction is given by a vector a of any length, then

$D_a f = \frac{df}{ds} = \frac{1}{|a|}a.\text{grad } f$ (where $\frac{a}{|a|}$ is a unit vector is direction of a)

ILLUSTRATIVE EXAMPLES

Example:

Gradient. Directional Derivative

Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point P: (2, 1, 3) in the direction of the vector $a = i - 2k$.

Solution:

We obtain grad f = 4xi + 6yj + 2zk, and at P: (2, 1, 3), grad f = 8i + 6j + 6k

$$D_a f = \frac{a}{|a|}\cdot \text{grad } f$$

$$= \frac{1}{\sqrt{5}}(i - 2k).(8i + 6j + 6k) = \frac{1.8 - 2.6}{\sqrt{5}} = -\frac{4}{\sqrt{5}} = -1.789$$

The minus sign indicates that f decreases at P in the direction of a.

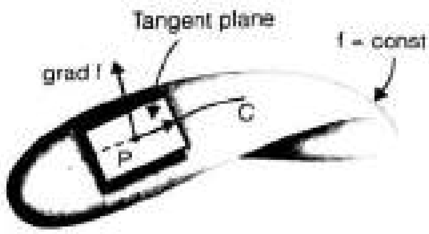
2.14.16 Gradient Characterizes Maximum Increase

Theorem. 1 (Gradient, Maximum Increase)

Let f(P) = f(x, y, z) be a scalar function having continuous first partial derivatives. Then grad f exists and its length and direction are independent of the particular choice of Cartesian coordinates in space. If at a point P the gradient of f is not the zero vector, it has the direction of maximum increase of f at P. Proof. From (6) and the definition of inner product we have

7. $$D_bf = |b||\text{grad } f|\cos\gamma = |\text{grad } f|\cos\gamma$$

where γ is the angle between b and grad f. Now f is a scalar function. Hence its value at a point P depends on P but not on the particular choice of coordinates. The same holds for the arc length s of the ray C (see hence also for D_bf. Now (7) shows that D_bf is maximum when $\cos\gamma = 1$, i.e. $\gamma = 0$, and the $D_bf = |\text{grad } f|$. It follows that the length and direction of grad f are independent of the coordinates. Since $\gamma = 0$ if and only if b has the direction of grad f, the latter is the direction of maximum increase of f at P, provided grad f $\neq$ 0 at P.



Gradient as surface normal vector

Gradient as Surface Normal Vector: Another basic use of the gradient results in connection with surfaces S in space given by

8. $$f(x, y, z) = c = \text{const.}$$

as follows. We recall that a curve C in space can be given by

9. $$r(t) = x(t)i + y(t)j + z(t)k$$

Now if we want C to lie on S, its components must satisfy (8); thus

10. $$f(x(t), y(t), z(t)) = c$$

A tangent vector of C is

$$r'(t) = x'(t)i + y'(t)j + z'(t)k.$$

If C lies on S, this vector is tangent to S. At a fixed point P on S, these tangent vectors of all curves on S through P will generally form a plane, called the tangent plane of S at P (Figure above). Its normal (the straight line through P and perpendicular to the tangent plane) s called the surface normal of S at P. A vector parallel to it is called a surface normal vector of S at P. Now if we differentiate (10) with respect to t, we get by the chain rule.

11. $$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = (\text{grad } f)r' = 0$$

This means orthogonality of grad f and all the vectors r' in the tangent plane. This result in shown pictorially in the figure above, where grad f is shown as normal to tangent plane of vectors r'. So, we have the theorem 2 given below.

Theorem. 2 (Gradient as Surface Normal Vector)

Let f be a differentiable scalar function that represents a surface S: f(x, y, z) = c = const. Then if the gradient of f at a point P of S is not the zero vector, it is a normal vector of S at P.

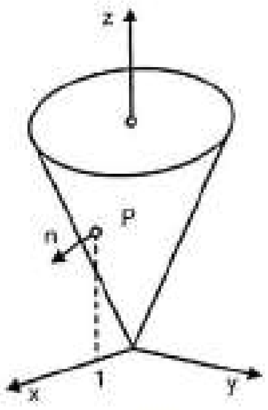
Comment. The surfaces given by (8) with various values of c are called the level surfaces of the scalar function f.

ILLUSTRATIVE EXAMPLES

Example:

Gradient as Surface Normal Vector

Find a unit normal vector n of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point P: (1, 0, 2).



Cone and unit normal vector n

Solution:

The cone is the level surface f = 0 f(x, y, z) = $4(x^2 + y^2) - z^2$, thus grad f = 8xi + 8yj − 2zk and at P(1, 0, 2), grad f = 8i − 4k

Hence, by Theorem 2, grad f is a normal vector of the cone at point P. Now a unit normal vector at point P will be,

$$n = \frac{1}{|\text{grad } f|}\text{grad } f = \frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}k$$

and the other one is −n.

2.14.17 Vector Fields that are Gradients of a Scalar Field ("Potential")

Some vector fields have the advantage that they can be obtained from scalar fields, which can be handled more easily. Such a vector field is given by a vector function v(P), which is obtained as the gradient of a scalar function, say, v(P) = grad f(P). The function f(P) is called a potential function or a potential of v(P). Such a v(P) and the corresponding vector field are called conservative because in such a vector fields, energy is conserved; that is, no energy is lost (or gained) in displacing a body (or a charge in the case of an electrical field) from a point P to another point in the field and back to P.

2.14.18 Divergence of a Vector Field

Vector calculus owes much of its importance in engineering and physics to the gradient, divergence, and curl. Having discussed the gradient, we turn next to the divergence. The curl follows in next section. Let v(x, y, z) be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of v. Then the function

1. $$\text{div } v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of v or the divergence of the vector field defined by v. Another common notation for the divergence of v is ∇.v,

$$\text{div } v = \nabla.v$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(v_1i + v_2j + v_3k) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

with the understanding that the "product" $(\partial/\partial x)v_1$ in the dot product means the partial derivative $\partial v_1/\partial x$, etc. This is a convenient notation, but nothing more. Note that ∇.v means the scalar div v, whereas, ∇f means the vector grad f.

ILLUSTRATIVE EXAMPLES

Example:

If $$v = 3xzi + 2xyj - yz^2k,$$
then $$\text{div } v = 3z + 2x - 2yz$$

We shall see below that the divergence has an important physical meaning. Clearly the values of a function that characterize a physical or geometrical property must be independent of the particular choice of coordinates; that is, those values must be invariant with respect to coordinate transformations.

Theorem. 1 (Invariance of The Divergence)

The values of div v depend only on the points in space (and, of course, on v) but not on the particular choice of the coordinates.

Now, let us turn to the more immediate practical task of getting a feel for the significance of the divergence.

If f(x, y, z) is a twice differentiable scalar function, then

$$\text{grad } f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

and $$\text{div (grad f)} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. The expression on the right is the Laplacian of f. Thus

$$\text{div (grad f)} = \nabla^2 f.$$

ILLUSTRATIVE EXAMPLES

Example: 1

Gravitational force

The gravitational force p, is the gradient of the scalar function f(x, y, z) = c/r, which satisfies Laplace's equation $\nabla^2 f = 0$. According to (3), this means that div p = 0(r > 0)

The following example, taken from hydrodynamics, shows the physical significance of the divergence of a vector field (and more will be added in next section when the so-called divergence theorem of Gauss will be available).

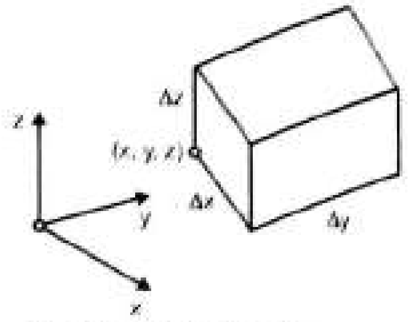
Example: 2

Motion of a compressible fluid. Physical meaning of the divergence

We consider the motion of a fluid in a region R having no sources or sinks in R, tht is, no points at which fluid is produced or disappears. The concept of fluid state is meant to cover also gases and vapors. Fluids in the restricted sense, or liquids (water or oil, for instance), have very small compressibility, which can be neglected in many problems. Gasses and vapors have large compressibility; that is, their density r(= mass per unit volume) depends on the coordinates x, y, z in space (and may depend on time t). We assume that our fluid is compressible.

We consider the flow through a small rectangular box W of dimensions Δx, Δy, Δz with edges, parallel to the coordinate axes (Fig. below). W has the volume ΔV = Δx Δy Δz. Let $v = [v_1, v_2\ v_3] = v_1i + v_2j + v_3k$ be the velocity vector of the motion. We set

2. $u = \rho v = [u_1, u_2, u_3] = u_1 i + u_2 j + u_3 k$ and assume that u and v are continuously differentiable vector functions of x, y, z, and t (that is, they have first partial derivatives, which are continuous). Let us calculate the change in the mass included in W by considering the flux across the boundary, that is, the total loss of mass leaving W per unit time. Consider the flow through the left face of W, whose area is $\Delta x\, \Delta z$. The components, v_1 and v_3 of v are parallel to that face and contribute nothing to this flow. Hence the mass of fluid entering through that face during a short time interval Δt is given approximately by



Physical interpretation of the divergence

$$(\rho v_2)_y \Delta x \Delta z \Delta t = (u_2)_y \Delta x\, \Delta z\, \Delta t,$$

where the subscript y indicates that this expression refers to the left face. The mass of fluid leaving the box W through the opposite face during the same time interval is approximately $(u_2)_{y+\delta y} \Delta x\, \Delta z\, \Delta t$, where the subscript $y + \Delta y$ indicates that this expression refers to the right face (which is not visible in Fig. above figure). The difference

$$\Delta u_2\, \Delta x\, \Delta z\, \Delta t = \frac{\Delta u_2}{\Delta_y} \Delta V\, \Delta t \qquad \left[\Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y\right]$$

is the approximate loss of mass. Two similar expressions are obtained by considering the other two pairs of parallel faces of W. If we add these three expression, we find that the total loss of mass in W during the time interval Δt is approximately.

$$\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)\Delta V\, \Delta t$$

where, $\Delta u_1 = (u_1)_{x+\Delta x} - (u_1)_x$

and $\Delta u_3 = (u_3)_{z+\Delta z} - (u_3)_z$

This loss of mass in W is cause by the time rate of change of the density and is thus equal to

$$-\frac{\Delta \rho}{\Delta t}\Delta V\, \Delta t$$

If we equate both expressions, divide the resulting equation by $\Delta V\, \Delta t$, we get

$$\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z} = -\frac{\Delta P}{\Delta t}$$

Now we let Δx, Δy, Δz and Δt approach zero & get,

$$\text{div } u = \text{div}(\rho v) = -\frac{\partial \rho}{\partial t}$$

3. i.e. $$\frac{\partial \rho}{\partial t} + \text{div}(\rho v) = 0$$

This important relation is called the condition for the conservation of mass or the continuity equation of a compressible fluid flow.

If the flow is steady, that is, independent of time, then $\frac{\partial \rho}{\partial t} = 0$ and the continuity equation is

4. $$\text{div}(\rho v) = 0$$

If the density r is constant, so that the fluid is incompressible, then equation (6) becomes

5. $$\text{div } v = 0$$

This relation is known as the condition of incompressibility. It expresses the fact that the balance of outflow and inflow for a given volume element is zero at any time. Clearly, the assumption that the flow has no source or sinks in R is essential to our argument.

From this discussion you should conclude and remember that, roughly speaking, the divergence measures outflow minus inflow.

If v denotes the velocity of fluid in a medium and if div(v) = 0, then the fluid is said to be **incompressible**. In electromagnetic theory, if div(v) = 0, then the vector field v is said to be **solenoidal**.

2.14.19 Curl of a Vector Field

Gradient, divergence, and curl are basic in connection with fields. We now define and discuss the curl. Let x, y, z be right-handed Cartesian coordinats, and let

$$v(x, y, z) = v_1 i + v_2 j + v_3 k$$

be a differentiable vector function. Then the function

$$\text{curl } v = \nabla \times v = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\text{curl } v = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k$$

is called the curl of the vector function v or the curl of the vector field defined by v.

Instead of curl v, the notation rot v is also used, (since one application of curl is to signify rotation of a rigid body)

ILLUSTRATIVE EXAMPLES

Example: 1

With respect to right-handed Cartesian coordinates, let

$$v = yzi + 3zxj + zk.$$

Then (1) gives

$$\text{curl } v = \nabla \times v$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$$

$$= -3xi + yj + (3z - z)k = -3xi + yj + 2zk.$$

The curl plays an important role in many applications. Let us illustrate this with a typical basic example. (We shall say more about the role and nature of the curl in next section).

Example: 2

Rotation of a rigid body. Relation to the curl

Rotation of a rigid body B about a fixed axis in space can be described by a vector w of magnitude ω in the direction of the axis of rotation, where ω(> 0) is the angular speed of the rotation, and ω is directed so that the rotation appears clockwise if we look in the direction of w. The velocity field of the rotation can be represented in the form

$$v = w \times r$$

where r is the position vector of a moving point with respect to a Cartesian coordinate system having the origin on the axis of rotation. Let us choose right-handed Cartesian coordinates such that

$$w = \omega k \text{ and } r = xi + yj + zk$$

that is, the axis of rotation is the z-axis. Then

$$v = w \times r = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y i + \omega x j$$

and therefore,

$$\text{curl } v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega k.$$

since $w = \omega k$,

2. $$\text{curl } v = 2w.$$

Hence, in the case of a rotation of a rigid body, the curl of the velocity field has the direction of the axis of rotation, and its magnitude equals twice the angular speed ω of the rotation.

Note that out result does not depend on the particular choice of the Cartesian coordinate system in space.

For any twice continuously differentiable scalar function f,

3. $$\text{curl (grad f)} = 0,$$

as can easily be verified by direct calculation, as shown below:

$$\text{grad f} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\text{curl(grad f)} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2}{\partial y \partial z} f\right) - j\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)$$

$$= 0i - 0j + 0k = 0$$

Hence if a vector function is the gradient of a scalar function, its curl is the zero vector. Since the curl characterizes the rotation in a field, we also say more briefly that gradient fields describing a motion are irrotational. (If such a field occurs in some other connection, not as a velocity field, it is usually called conservative;

If curl v = 0, then v is soild to be an irrotational field.

ILLUSTRATIVE EXAMPLES

Example:

The gravitational field has curl p = 0. The field in the rotation of rigid body example this section is not irrotational since we saw that curl v = 2w ≠ 0. A similar velocity field is obtained by stirring coffee in a cup.

Other than (3), another key formula for any twice continuously differentiable scalar function is

4. $$\text{div}(\text{curl } v) = 0$$

It is plausible because of the interpretation of the curl as a rotation and the divergence as a flux. A proof of (4) follows readily from the definitions of curl and div; the six terms cancel in pairs. Let

$$v = v_1 i + v_2 j + v_3 k$$

$$\text{curl } v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - j\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + k\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\text{div}(\text{curl } v) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$$= 0$$

The curl is defined in terms of coordinates, but if it is supposed to have a physical or geometrical significance, it should not depend on the choice of these coordinates. This is true, as follows.

Theorem. 1 (Invariance of The Curl)

The length and direction of curl v are independent of the particular choice of Cartesian coordinate systems in space.

2.14.19.1 Important Repeated Operations by Nable Operator (∇)

1. div grad f = $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
2. curl grad f = $\nabla \times \nabla f = 0$
3. div curl f = $\nabla \cdot (\nabla \times F) = 0$
4. curl curl f = grad div F − $\nabla^2 F = \nabla(\nabla \cdot F) - \nabla^2 F$
5. grad div f = curl curl F + $\nabla^2 F = \nabla \times \nabla \times F + \nabla^2 \times F$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.156 The directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point P(2, 1, 3) in the direction of the vector a = i − 2 k is

(a) −2.785 (b) −2.145

(c) −1.789 (d) 1.000

[CE, GATE-2006, 2 marks]

Solution: (c)

$$f = 2x^2 + 3y^2 + z^2, \; P(2, 1, 3), \; a = i - 2k$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 4xi + 6yj + 2zk$$

at P (2, 1, 3) $\nabla f = 4 \times 2 \times i + 6 \times 1 \times j + 2 \times 3 \times k = 8i + 6j + 6k$

directional derivative of f in direction of vector

$$a = i - 2k \text{ is}$$

nothing but the component of gradf in the direction of vector a and is given by $\frac{a}{|a|}$ grad f

$$= \left(\frac{i - 2k}{\sqrt{1^2 + (-2)^2}}\right) \cdot (8i + 6j + 6k)$$

$$= \frac{1}{\sqrt{5}}(1.8 + 0.6 + (-2)6) = \frac{-4}{\sqrt{5}} = -1.789$$

Q.157 A velocity vector is given as $\vec{V} = 5xy\vec{i} + 2y^2\vec{j} + 3yz^2\vec{k}$. The divergence of this velocity vector at (1, 1, 1) is

(a) 9
(b) 10
(c) 14
(d) 15

[CE, GATE-2007, 2 marks]

Solution: (d)

$$\vec{V} = 5xyi + 2y^2j + 3yz^2k = v_1i + v_2j + v_3k$$

$$\text{div}(\vec{V}) = \frac{dv_1}{dx} + \frac{dv_2}{dy} + \frac{dv_3}{dz} = 5y + 4y + 6yz$$

at (1, 1, 1) $\text{div}(\vec{V}) = 5.1 + 4.1 + 6.1.1 = 15$

Q.158 Potential function ϕ is given as $\phi = x^2 - y^2$. What will be the stream function (ψ) with the condition $\psi = 0$ at $x = y = 0$?

(a) $2xy$
(b) $x^2 + y^2$
(c) $x^2 - y^2$
(d) $2x^2y^2$

[CE, GATE-2007, 2 marks]

Solution: (a)

Stream function, $\psi = 2xy$

Q.159 For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the gradient at the point P(1, 2, – 1) is

(a) $2\vec{i} + 6\vec{j} + 4\vec{k}$
(b) $2\vec{i} + 12\vec{j} - 4\vec{k}$
(c) $2\vec{i} + 12\vec{j} + 4\vec{k}$
(d) $\sqrt{56}$

[CE, GATE-2009, 1 mark]

Solution: (b)

$$f = x^2 + 3y^2 + 2z^2$$

$$\Delta f = \text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2x) + j(6y) + k(4z)$$

The gradient at P(1, 2, –1) is

$$= i(2 \times 1) + j(6 \times 2) + k(4 \times -1)$$

$$= 2i + 12j - 4k$$

Q.160 For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the directional derivative at the point $P(1, 2, -1)$ in the direction of a vector $\vec{i} - \vec{j} + 2\vec{k}$ is

(a) -18
(b) $-3\sqrt{6}$
(c) $3\sqrt{6}$
(d) 18

[CE, GATE-2009, 2 marks]

Solution: (b)

$$\Delta f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

here $f = x^2 + 3y^2 + 2z^2$

$\therefore$ $\Delta f = i(2x) + j(6y) + k(4z)$

at $p(1, 2, -1)$ $\Delta f = i(2 \times 1) + j(6 \times 2) + k(4 \times -1) = 2i + 12j - 4k$

The directional derivative in direction of vector $a = i - j + 2k$ is given by

$$\frac{a}{|a|}.\text{grad } f = \frac{i - j + 2k}{\sqrt{1^2 + (-1)^2 + 2^2}}.(2i + 12j - 4k)$$

$$= \frac{1}{\sqrt{6}}(1.2 + (-1).12 + 2(-4)) = -\frac{18}{\sqrt{6}} = -3\sqrt{6}$$

Q.161 The directional derivative of the field $u(x, y, z) = x^2 - 3yz$ in the direction of the vector $(\bar{i} + \bar{j} - 2\bar{k})$ at point $(2, -1, 4)$ is ________.

[CE, GATE-2015 : 2 Marks, Set-I]

Solution:

$$u(x, y, z) = x^2 - 3yz$$

$$\nabla u = 2x\hat{i} - 3zj - 3y\hat{k}$$

$$\nabla u|_{\text{at }(2, -1, 4)} = 4\hat{i} + 12\hat{j} - 3\hat{k}$$

Directional derivative, $= (4\hat{i} + 12\hat{j} - 3\hat{k})\cdot\frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}} = \frac{4 - 12 - 6}{\sqrt{6}} = -\frac{14}{\sqrt{6}}$

$$= -\frac{7\sqrt{6}}{3} = -5.715$$

Q.162 Equation of the line normal to function $f(x) = (x-8)^{2/3} + 1$ at P(0, 5) is

(a) $y = 3x - 5$
(b) $y = 3x + 5$
(c) $3y = x + 15$
(d) $3y = x - 15$

[ME, GATE-2006, 2 marks]

Solution: (b)

Given $f(x) = (x - 8)^{2/3} + 1$

$$f'(x) = \frac{2}{3}(x - 8)^{-1/3}$$

Slope of tangent at point (0, 5) $m = \frac{2}{3}(0 - 8)^{-1/3} = -\frac{1}{3}$

Slope of normal at point (0, 5) $m_1 = -\frac{1}{m} = 3$

Equation of normal at point (0, 5)

$$y - 5 = 3(x - 0)$$

$$\Rightarrow \quad y = 3x + 5$$

Q.163 The divergence of the vector field $(x - y)\hat{i} + (y - x)\hat{j} + (x + y + z)\hat{k}$ is

(a) 0
(b) 1
(c) 2
(d) 3

[ME, GATE-2008, 1 mark]

Solution: (d)

$$\text{div } [(x-y)\hat{i} + (y-x)\hat{j} + (x+y+z)\hat{k}] = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-x) + \frac{\partial}{\partial z}(x+y+z) = 3$$

Q.164 The directional derivative of the scalar function $f(x, y, z) = x^2 + 2y^2 + z$ at the point $P = (1, 1, 2)$ in the direction of the vector $\vec{a} = 3\hat{i} - 4\hat{j}$ is

(a) −4
(b) −2
(c) −1
(d) 1

[ME, GATE-2008, 2 marks]

Solution: (b)

$$\text{grad } f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = 2xi + 4yj + k$$

at point $P(1, 1, 2)$, grad $f = 2i + 4j + k$

Now directional derivative of f at P(1, 1, 2) in direction of vector $a = 3i - 4j$ is given by

$$\frac{a}{|a|} \text{ grad } f = \left(\frac{3i - 4j}{\sqrt{25}}\right) \cdot (2i + 4j + k)$$

$$= \frac{1}{5}(3.2 - 4.4 + 0) = -2$$

Q.165 The divergence of the vector field $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at a point (1, 1, 1) is equal to

(a) 7
(b) 4
(c) 3
(d) 0

[ME, GATE-2009, 1 mark]

Solution: (c)

Vector field, $\vec{f} = 3xz\,\hat{i} + 2xy\,\hat{j} - yz^2\,\hat{k} = v_1 i + v_2 j + v_3 k$

Divergence of vector field

$$\text{Div (f)} = \nabla . f = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \frac{\partial}{\partial x}[3xz] + \frac{\partial}{\partial y}[2xy] + \frac{\partial}{\partial z}[-2yz^2] = 3z + 2x - 2zy$$

$$\text{Div (f)}\big|_{(1, 1, 1)} = 3(1) + 2(1) - 2(1)(1) = 3$$

Q.166 Velocity vector of a flow field is given as $\vec{V} = 2xy\hat{i} - x^2z\hat{j}$. The vorticity vector at (1, 1, 1) is

(a) $4\hat{i} - \hat{j}$
(b) $4\hat{i} - \hat{k}$
(c) $\hat{i} - 4\hat{j}$
(d) $\hat{i} - 4\hat{k}$

[ME, GATE-2010, 2 marks]

Solution: (d)

$$\text{Velocity vector} = \vec{V} = 2xy\hat{i} - x^2z\hat{j}$$

$$\text{The vorticity vector} = \text{curl (velocity vector)}$$

$$= \text{curl}(\vec{V})$$

$$= \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -x^2z & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-x^2z)\right]\hat{i} - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2xy)\right]\hat{j} + \left[\frac{\partial}{\partial x}(-x^2z) - \frac{\partial}{\partial y}(2xy)\right]\hat{k}$$

$$= x^2\hat{i} + [-2xz - 2x]\hat{k}$$

at (1, 1, 1), by substituting $x = 1$, $y = 1$ and $z = 1$,

we get, $\quad$ vorticity vector $= \hat{i} - 4\hat{k}$

Q.167 For the spherical surface $x^2 + y^2 + z^2 = 1$, the unit outward normal vector at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is given by

(a) $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(b) $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

(c) $\hat{k}$

(d) $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

[ME, GATE-2012, 1 mark]

Solution: (a)

$$x^2 + y^2 + z^2 = 1$$

$$f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \quad \text{at}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{grad } f = \frac{2}{\sqrt{2}}\hat{i} + \frac{2}{\sqrt{2}}\hat{j} + 2\times 0\times\hat{k} = \sqrt{2}\,i + \sqrt{2}\,j + 0\hat{k}$$

$$|\text{grad } f| = \sqrt{2+2} = \sqrt{4} = 2$$

The unit outward normal vector at point P is

$$n = \frac{1}{|\text{grad } f|}(\text{grad } f)_{\text{at P}} = \frac{1}{2}\left(\sqrt{2}\,i + \sqrt{2}\,j\right) = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

Q.168 Curl of vector $V(x, y, z) = 2x^2i + 3z^2j + y^3k$ at $x = y = z = 1$ is

(a) $-3i$

(b) $3i$

(c) $3i - 4j$

(d) $3i - 6k$

[ME, GATE-2015 : 1 Mark, Set-2]

Solution: (a)

$$\text{Curl of vector} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 & 3z^2 & y^3 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(y^3) \frac{\partial}{\partial z}(3z^2)\right] - j\left[\frac{\partial}{\partial x}(y^3) \frac{\partial}{\partial z}(2x^2)\right] + k\left[\frac{\partial}{\partial x}(3z^2) \frac{\partial}{\partial y}(2x^2)\right]$$

$$= i[3y^2 - 6z] - j[0] + k[0 + 0]$$

At $x = 1$, $y = 1$ and $z = 1$

Curl $= i(3 \times 1^2 - 6 \times 1) = -3i$

Q.169 Let ϕ be an arbitrary smooth real valued scalar function and V be an arbitrary smooth vector valued function in a three-dimensional space. Which one of the following is an identity?

(a) $Curl(\phi\vec{V}) = \nabla(\phi Div\vec{V})$

(b) $Div\vec{V} = 0$

(c) $Div\ Curl\vec{V} = 0$

(d) $Div\left(\phi\vec{V}\right) = \phi Div\vec{V}$

[ME, GATE-2015 : 1 Mark, Set-3]

Solution: (c)

$$Div\ Curl\vec{V} = 0$$

∴ (c) is correct option.

Q.170 For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, magnitude of the gradient at the point (1, 3) is

(a) $\sqrt{\frac{13}{9}}$

(b) $\sqrt{\frac{9}{2}}$

(c) $\sqrt{5}$

(d) $\frac{9}{2}$

[EE, GATE-2005, 2 marks]

Solution: (c)

$$u = \frac{x^2}{2} + \frac{y^2}{3}$$

$$\text{grad } u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} = xi + \frac{2}{3}yj$$

At(1,3), $$\text{grad } u = (1)i + \left(\frac{2}{3}\cdot 3\right)j = i + 2j$$

$$|\text{grad } u| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Q.171 Divergence of the three-dimensional radial vector field $\vec{r}$ is

(a) 3

(b) 1/r

(c) $\hat{i} + \hat{j} + \hat{k}$

(d) $3\left(\hat{i} + \hat{j} + \hat{k}\right)$

[EE, GATE-2010, 1 mark]

Solution: (a)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{div } \vec{r} = \nabla . \vec{r}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . (x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$
$$= 1 + 1 + 1 = 3$$

Q.172 The curl of the gradient of the scalar field defined by $V = 2x^2y + 3y^2z + 4z^2x$ is

(a) $4xy\,\mathbf{a}_x + 6yz\,\mathbf{a}_y + 8zx\,\mathbf{a}_z$
(b) $4\mathbf{a}_x + 6\mathbf{a}_y + 8\mathbf{a}_z$
(c) $(4xy + 4z^2)\,\mathbf{a}_x + (2x^2 + 6yz)\,\mathbf{a}_y + (3y^2 + 8zx)\,\mathbf{a}_z$
(d) 0

[EE, GATE-2013, 1 Mark]

Solution: (d)

Curl of gradient of a scalar field is always zero.

$$\nabla \times \nabla V = 0$$

Q.173 $\nabla \times \nabla \times P$, where P is a vector is equal to

(a) $P \times \nabla \times P - \nabla^2 P$
(b) $\nabla^2 P + \nabla(\nabla \times P)$
(c) $\nabla^2 P + \nabla \times P$
(d) $\nabla(\nabla \cdot P) - \nabla^2 P$

[EC, GATE-2006, 1 mark]

Solution: (d)

From property of vector triple product.

$$A \times (B \times C) = (A.C)\,B - (A.B)C$$

and putting, $A = \nabla$, $B = \nabla$ & $C = P$

We get, $$\nabla \times \nabla \times P = (\nabla P)\nabla - (\nabla.\nabla)P = \nabla(\nabla P) - \nabla^2 P$$

Q.174 The divergence of the vector field $\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is

(a) 0
(b) 1/3
(c) 1
(d) 3

[EC, GATE-2013, 1 Mark]

Solution: (d)

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$$
$$\nabla \cdot \vec{A} = 3$$

Q.175 Function f is known at the following points

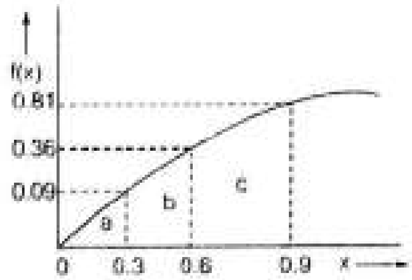
x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0
f(x)	0	0.09	0.36	0.81	1.44	2.25	3.24	4.41	5.76	7.29	9.00

The value of $\int_0^3 f(x)dx$ computed using the continuous at x = 3?

(a) 8.983
(b) 9.003
(c) 9.017
(d) 9.045

[CS, GATE-2013, 1 Mark]

Solution: (d)



Area of region a is

$$= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 0.09 \times 0.3$$

Area of region b is

$$= \frac{1}{2} \times \text{height}*(\text{base1}+\text{base2}) = \frac{1}{2} \times 0.3 \times (0.09 * 0.36)$$

$$\int_0^3 f(x)dx = \frac{1}{2}(0.3) \times (0.09) + \frac{1}{2}(0.3) \times (0.09 + 0.36) + \cdots + \frac{1}{2}(0.3) \times (7.29 + 9.0) = 9.045$$

option (d) is correct.

Q.176 The direction of vector A is radially outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla \cdot A = 0$ is

(a) –2
(b) 2
(c) 1
(d) 0

[IN, GATE-2012, 2 marks]

Solution: (a)

$$|A| = kr^n$$

$$\Rightarrow \quad A = kr^n \frac{\bar{r}}{r}$$

$$\nabla \cdot A = \nabla \cdot (kr^{n-1}\bar{r}) = 0$$

We have, $\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi(\nabla \cdot A)$

$$K\left[\nabla(r^{n-1}) \cdot \bar{r} + r^{n-1}(\nabla \cdot \bar{r})\right] = 0$$

$$K\left[(n-1)r^{n-2}\frac{\bar{r}}{r} \cdot \bar{r} + 3r^{n-1}\right] = 0$$

$$(n-1)\, r^{n-3} r^2 + 3r^{n-1} = 0$$

$$[(n-1)+3]r^{n-1} = 0$$

$$n = -2$$

Q.177 For a vector E, which one of the following statements is NOT TRUE?

(a) If $\nabla \cdot E = 0$, E is called solenoidal.
(b) If $\nabla \times E = 0$, E is called conservative.
(c) If $\nabla \times E = 0$, E is called irrotational.
(d) If $\nabla \cdot E = 0$, E is called irrotational.

[IN, GATE-2013 : 1 mark]

Solution: (d)

Option (d) is not true as irrotational vector has cross product as zero. Thus for vector to be irrotational $\nabla \times E = 0$

Q.178 The magnitude of the directional derivative of the function $f(x, y) = x^2 + 3y^2$ in a direction normal to the circle $x^2 + y^2 = 2$, at the point (1, 1), is

(a) $4\sqrt{2}$ (b) $5\sqrt{2}$
(c) $7\sqrt{2}$ (d) $9\sqrt{2}$ **[IN, GATE-2015 : 1 Mark]**

Solution: (a)

$$f(x, y) = x^2 + 3y^2$$
$$\phi = x^2 + y^2 - 2 \quad \text{and point} \quad P \Rightarrow (1, 1)$$

Normal to the surface,

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} = 2x\hat{i} + 2y\hat{j}$$

$$\nabla\phi|_{\text{at } P(1,1)} = 2\hat{i} + 2\hat{j}$$

the normal vector is $\vec{a} = 2\hat{i} + 2\hat{j}$

Magnitude of directional derivative of f along $\hat{a}$ at (1, 1) is $\Rightarrow \nabla f \cdot \hat{a}$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} = 2x\hat{i} + 6y\hat{j}$$

$$\nabla f|_{(1,1)} = 2\hat{i} + 6\hat{j}$$

$$|\vec{a}| = \sqrt{4+4} = 2\sqrt{2}$$

$$\hat{a} = \frac{2\hat{i} + 2\hat{j}}{2\sqrt{2}} = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$$

∴ Magnitude of directional derivative

$$= (2\hat{i} + 6\hat{j})\left(\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right) = \frac{2+6}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

Q.179 Which one of the following is a property of the solutions to the Laplace equation: $\nabla^2 f = 0$?

(a) The solutions have neither maxima nor minima anywhere except at the boundaries.
(b) The solutions are not separable in the coordinates.
(c) The solutions are not continuous.
(d) The solutions are not dependent on the boundary conditions.

[EC, 2016 : 1 Mark, Set-1]

Answer: (a)

2.14.20 Vector Integral Calculus: Integral Theorems

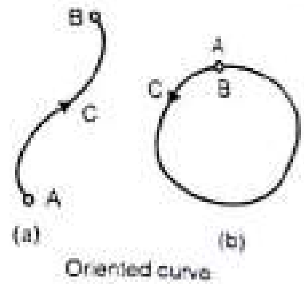
2.14.20.1 Line Integral

The concept of a line integral is a simple and natural generalization of a definite integral

1. $\int_a^b f(x)\,dx$ known from calculus. In (1) we integrate the integrand f(x) from x = a along the x-axis to x = b. In a line integral we shall integrate a given function, called the integrand, along a curve C in space (or in the plane). Hence curve integral would be a better term, but line integral is standard. We represent the curve C by a parametric representation.

2. $r(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k \quad (a \le t \le b)$

We call C the path of integrating. A : r(a) its initial point, and B: r(b) its terminal point. C is now oriented. The direction from A to B, in which t increases, is called the positive direction on C. We can indicate the direction by an arrow (as in above Figure (a)). The points A and B may coincide (as in above figure (b)). Then C is called a closed path.



Oriented curve

We call C a smooth curve if C has a unique tangent at each of its points whose direction varies continuously as we move along C. Technically : C has a representation (2) such that r(t) is differentiable and the derivative r′(t) = dx/dt is continuous and different from the zero vector at every point of C.

2.14.20.2 Definition and Evaluation of Line Integrals

A line integral of a vector function F(r) over a curve C is defined by

3. $$\int_C F(r)\cdot dr = \int_a^b F(r(t))\cdot\frac{dr}{dt}dt$$

In terms of components, with dr = [dx, dy, dz] and = d/dt, formula (3) becomes

3′. $$\int_C F(r)\cdot dr = \int_C (F_1 i + F_2 j + F_3 k).(dx i + dy\, j + dz\, k)$$

$$= \int_C F_1 dx + F_2 dy + F_3 dz = \int_a^b (F_1 x' + F_2 y' + F_3 z')$$

If the path of integrating C in (3) is a closed curve, then instead of

$$\int_C \text{ we also write } \oint_C .$$

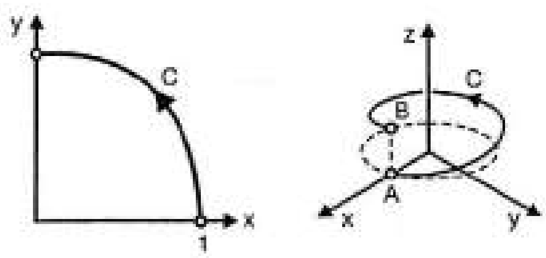
We see that the integral in (3) on the right is a definite integral of a function of t taken over the interval $a \le t \le b$ on the t-axis in the positive direction (the direction of increasing t). This definite integral exists for continuous F and piecewise smooth C, because this makes F.r′ piecewise continuous.

ILLUSTRATIVE EXAMPLES

Example: 1

Find the value of the line integral (3) when $F(r) = [-y, -xy] = -yi - xyj$ and C is the circular arc as from A to B shown in figure titled Example below:

Solution:



We may represent C by

$$r(t) = [\cos t, \sin t] = \cos ti + \sin tj \ (0 \le t \le \pi/2)$$

Thus $x(t) = \cos t,\ y(t) = \sin t$, so that

$$F(r(t)) = -y(t)i - x(t)y(t)j = [-\sin t, -\cos t \sin t] = -\sin t\, i - \cos t \sin t\, j$$

By differentiation, $r'(t) = -\sin t\, i + \cos t\, j$.

So by (3)

$$\int_C F(r).dr = \int_a^b F(r(t))\cdot\frac{dr}{dt}dt = \int_0^{\pi/2}(-\sin t i - \cos t \sin t j).(-\sin t i + \cos t j)dt$$

$$= \int_0^{\pi/2}(\sin^2 t - \cos^2 t \sin t)dt$$

$$= \int_0^{\pi/2}\sin^2 t dt - \int_0^{\pi/2}\cos^2 t \sin t dt$$

$$= \int_0^{\pi/2}\left(\frac{1-\cos^2 t}{2}\right)dt + \int^0 u^2\, du \quad \text{(where } u = \cos t)$$

$$= \left(\frac{\pi}{4} - 0\right) - \left(\frac{1}{3}\right) \simeq 0.4521$$

Example: 2

Line integral in space.

Evaluation of line integrals in space is practically the same as it is in the plane. To see this, find the value of (3) when $\int F(r)dr = [z, x, y] = zi + xj + yk$ and C is the helix (Figure above titled Example 2) $r(t) = [\cos t, \sin t, 3t]$ where $0 \le t \le 2\pi$.

Solution:

We have $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 3t$. Thus

$$F(r) = zi + xj + yk = 3ti + \cos tj + \sin tk$$

$$\int F(r(t))dr = \int F(r(t)).r'(t)dt$$

$$= \int_0^{2\pi}(3ti + \cos tj + \sin tk).(-\sin ti + \cos tj + 3k)dt$$

$$= \int_0^{2\pi}(-3t\sin t + \cos^2 t + 3\sin t)dt$$

$$= 6\pi + \pi + 0 = 7\pi$$

$$\approx 21.99.$$

1. **Choice of representation** : Does the value of a line integral with given F and C depend on the particular choice of a representation of C? The answer is no; see theorem 1 below.
2. **Choice of path** : Does this value change if we integrate from the old point A to the old point B but along another path. The answer is yes, in general; see example 3.

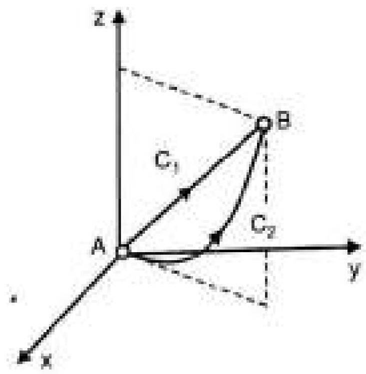
Example: 3

Dependence of a line integral on path (same endpoints)

Evaluate the line integral (3) with $F(r) = [5z, xy, x^2z] = 5zi + xyj + x^2zk$ along two different paths with the same initial point A:(0, 0, 0) and the same terminal point B:(1, 1, 1), namely (Fig. below titled example 3)

(a) C_1 : the straight-line segment $r_1(t) = [t, t, t] = ti + tj + tk$, $0 \le t \le 1$, and

(b) C_2 : the parabolic are $r_2(t) = [t, t, t^2] = ti + tj + t^2k$, $0 \le t \le 1$.



Solution:

(a) By substituting r_1 into F we obtain $F(r1(t)) = [5t, t^2, t^3] = 5ti + t^2j + t^3k$. We also need

$$r'_1 = [1,1,1] = i + j + k.$$

Hence the integral over C_1 is

$$\int_{C_1} F(r).dr = \int_0^1 F(r_1(t)) \cdot r_1' dt = \int_0^1 (5ti + t^2 j + t^2 k).(i + j + k)dt$$

$$= \int_0^1 (5t + t^2 + t^3)dt = \frac{5}{2} + \frac{1}{3} + \frac{1}{4} = \frac{31}{12}$$

(b) Similarly, by substituting r_2 into F and calculating r'_2 we obtain for the integral over the path C_2.

$$\int_{C_2} F(r) \cdot dr = \int_0^1 F(r_2(t)).r_2'(t)dt = \int_0^1 (5t^2 + t^2 + 2t^5)dt = \frac{5}{3} + \frac{1}{3} + \frac{2}{6} = \frac{28}{12}.$$

The two results are different, although the endpoints are the same. This shows that the value of a line integral (3) will in general depend not only on F and on the endpoints A, B of the path but also on the path along wihch we integrate from A to B.

Can we find conditions that guarantee independence? This is a basic question in connection with physical applications. The answer is yes, as we show in next section.

2.14.20.3 General Properties of the Line Integral (3)

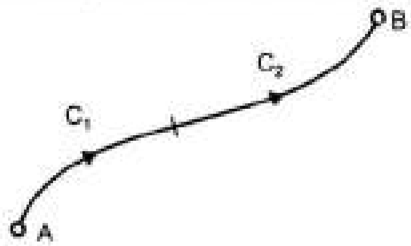
From familiar properties of integrals in calculate we obtain corresponding formulas for line integrals.

$$\int_C kF.dr = k\int_C F.dr \qquad \text{(k constant)}$$

$$\int_C (F + G).dr = \int_C F.dr + \int_C G.dr$$

$$\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr$$

where in third formula above the path C is subdivided into two arcs C_1 and C_2, that have the same orientation as C (Fig. below). In (second formula above) the orientation of C is the same in both integrals. If the sense of integration along C is reversed, the value of the integral is multiplied by −1.



2.14.20.4 Line Integrals Independent of Path

1. $$\int_C F(r).dr = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

as before. In (1) we integrate from a point A to a point B over a path C. The value of such an integral generally depends not only on A and B, but also on the path C along which we integrate. This was shown in example 3 of the last section. It raises the question of conditions for independence of path, so that we get the same value in integrating from A to B along any path C. This is of great practical importance. For instance, in mechanics, independence of path may mean that we have to do the same amount of work regardless of the path to the mountaintop, be it short and steep or long and gentle, or that we gain back the work done in extending an elastic spring when we release it. Not all forces are of this type - think of swimming in a big whirlpool.

We define a line integral (1) to be independent of path in a domain D in space if for every pair of endpoints A, B in D the integral (1) has the same value for all path in D that begin at A and end at B.

A very practical criterion for path independence is the following.

Theorem. 1 (Independence of Path)

A line integral (1) with continuous F_1, F_2, F_3 in a domain D in space is independent of path in D if and only if $F = [F_1, F_2, F_3]$ is the gradient of some function f in D.

2. $$F = \text{grad } f;$$

in components,

2′. $$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

ILLUSTRATIVE EXAMPLES

Example: 1

Independence of path. Show that the integral

$$\int_C F \cdot dr = \int_C (2xdx + 2ydy + 4zdz)$$

is independent of path in any domain in space and find its value if C has the initial point A: (0, 0, 0) and terminal point B: (2, 2, 2).

Solution:

By inspection we find that

$$F = [2x, 2y, 4z] = 2xi + 2yj + 4zk = \text{grad } f,$$

where $$f = x^2 + y^2 + 2z^2.$$

(If F is more complicated, proceed by integration, as in Example 2, below.) Theorem 1 now implies independence of path. To find the value of the integral, we can choose the convenient straight path

$$C: \quad r(t) = [t, t, t] = t(i + j + k), \quad 0 \le t \le 2.$$

and get r', $= i + j + k$; thus $F \cdot r' = 2t + 2t + 4t = 8t$ and from this

$$\int_C (2xdx + 2ydy + 4zdz) = \int_0^2 F \cdot r' dt = dt \int_0^2 8t \, dt = 16$$

Proof of Theorem 1:

1. Let (2) hold for some function f in D. Let C be any path in D from any point A to any point B, given by

$$r(t) = x(t)i + y(t)j + z(t)k, \qquad a \le t \le b$$

by chain rule, we get

$$\int_A^B (F_1dx + F_2dy + F_3dz) = \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt$$

$$= \int_a^b \frac{df}{dt}dt = f[x(t), y(t), z(t)]\Big|_{t=a}^{t=b} = f(B) - f(A)$$

This shows that the value of the integral is simply the difference of the values of f at the two endpoints of C and is, therefore, independent of the path C.

2. The converse proof of this theorem, that independence of path implies that F is gradient of some function f, is more complicated and not given here.

The above example 1 can, now be solved more easily as

$$\int_C F\,dr = f(B) - f(A) = f(2, 2, 2) - f(0, 0, 0)$$

$$= (2^2 + 2^2 + 2.2^2) - (0^2 + 0^2 + 2.0^2) = 16$$

An easy way of solving this problem follows from proof of theory 1, shown below:

The last formula in part (a) of the proof,

$$\int_A^B (F_1dx + F_2dy + F_3dz) = f(B) - f(A) \qquad [F = \text{grad } f]$$

is the analog of the usual formula for definite integrals in calculus.

$$\int_a^b g(x)dx = G(x)\Big|_a^b = G(b) - G(a) \qquad [G'(x) = g(x)]$$

3. **Potential theory** relates to our present discussion, if we remember, that f is called a potential of F = grad f. Thus the integral (1) is independent of path in D if and only if F is the gradient of a potential in D.

Example: 2

Independence of path. Determination of a potential

Evaluate the integral

$$I = \int_C (3x^2dx + 2yzdy + y^2dz)$$

from A: (0, 1, 2) to B: (1, – 1, 7) by showing that F has a potential and applying line integral formula.

Solution:

If F has a potential f, we should have

$f_x = F_1 = 3x^2$, $\quad f_y = F_2 = 2yz$, $\quad f_z = F_z = y^2$

We show that we can satisfy these conditions. By integration and differentiation.

$f = x^3 + g(y, z)$, $\Rightarrow$ $f_y = g_y = 2yz$, $\Rightarrow$ $g = y^2z + h(z)$

$f = x^3 + g(y, z)$ $\Rightarrow$ $f_z = g_z = y^2 + h'$, Now from first step we renow that, $f_z = y^2$, $\Rightarrow$ $g = y^2 z + 0 = y^2 z$

$\therefore\ y^2 + h^{\prime} = y^2 \quad \Rightarrow \quad h' = 0, \quad \Rightarrow \quad h = \text{constant} = 0$ (say)

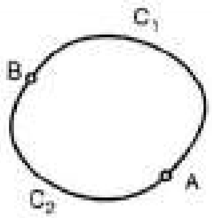
This gives $f(x, y, z) = x^3 + y^2 z$ and the required integral $I = f(B) - f(A)$

$$I = f(1, -1, 7) - f(0, 1, 2) = (1 + 7) - (0 + 2) = 6$$

Theorem. 2 (Independence of path)

The integral (1) is independent of path in a domain D if and only if its value around every closed path in D is zero.

Proof: If we have independence of path, integration from A to B along C_1 and along C_2 in Fig. 205 gives the same value. Now C_1 and C_2 together make up a closed curve C, and if we integrate from A along C_1 to B as before, but then in the opposite sense along C_2 back to A (so that this integral is multiplied by −1), the sum of the two integrals is zero, but this is the integral around the closed curve C.



Proof of Theorem 2

Conversely, assume that the integral around any closed path C in D is zero. Given any points A and B and any two curves C_1 and C_2 from A to B in D, we see that C_1 with the orientation reversed and C_1 together form a closed path C. By assumption, the integral over C is zero. Hence the integrals over C_1 and C_2. Both taken from A to B, must be equal. This proves the theorem.

Work. Conservative and Nonconservative (Dissipative) Physical Systems: Recall from the last section that in mechanics, the integral $\int_C F(r) \cdot dr$ represents the work done by a force F in the displacement of a body along C. Then theorem 2 states that work is independent of path if and only if it is zero for displacement around any closed path. Furthermore, Theorem 1 tells us that this happens if and only if F is the gradient of a potential. In this case, F and the vector field defined by F are called conservative, because in this case mechanical energy is conserved, that is, no work is done in the displacement from a point A and back to A. Similarly for the displacement of an electrical charge (an electron, for instance) in an electrostatic field.

Physically, the kinetic energy of a body can be interpreted as the ability of the body to do work by virtue of its motion, and if the body moves in a conservative field of force, after the completion of a round-trip the body will return to its initial position with the same kinetic energy it had originally. For instance, the gravitational force is conservative; if we throw a ball vertically up, it will (if we assume air resistance to be negligible) return to our hand with the same kinetic energy it had when it left our hand.

Friction, air resistance, and water resistance always act against the direction of motion, tending to diminish the total mechanical energy of a system (usually converting it into heat or mechanical energy of the surrounding medium, or both), and if in the motion of a body, these forces are so large that they can no longer be neglected, then the resultant F of the forces acting on the body is no longer conservative. Quite generally, a physical system is called conservative, if all the forces acting in it are conservative; otherwise it is called nonconservative or dissipative.

Exactness and Independence of Path: Theorem 1 relates path independence of the line integral (1) to the gradient and theorem 2 to integration around closed curves. A third idea and theorem 3, below) relate path independence to the exactness of the differential form

4. $$F_1dx + F_2dy + F_3dz$$

under the integral sign in (1). This form (4) is called exact in a domain D in space if it is the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

of a differentiable function f(x, y, z) everywhere in D. that is, if we have

$$F_1dx + F_2\,dy + F_3dz = df$$

Comparing these two formulas, we see that the form (4) is exact if and only if there is a differentiable function f(x, y, z) in D such that everywhere in D,

5. $$F_1 = \frac{\partial f}{\partial x} \qquad F_2 = \frac{\partial f}{\partial y}, \qquad F_3 = \frac{\partial f}{\partial z}$$

In vectorial form these three equation (5') can be written

5'. $$F = \text{grad } f.$$

Hence, by Theorem 1, the integral (1) is independent of path in D if and only if the differential form (4) has continuous components F_1, F_2, F_3 and is exact in D.

This is practically important because there is a useful exactness criterion involving the following concept.

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

For example, the interior of a sphere or a cube, the interior of a sphere with finitely many points removed, and the domain between two concentric spheres are simply connected, while the interior of a torus (a doughnut) and the interior of a cube with one space diagonal removed are not simply connected.

The criterion for path independence based on exactness is then as follows.

Theorem. 3 (Criterion for exactness and independence of path)

Let F_1, F_2, F_3 in the line integral,

$$\int_C F(r)\cdot dr = \int_C (F_1dx + F_2dy + F_3dz)$$

be continuous and have continuous first partial derivatives in a domain D in space. Then:

(a) If this integral is independent of path in D—and thus the differential form under the integral sign is exact—then in D,

6. $$\text{curl } F = 0$$

in components therefore condition of exactness follows from curl F = 0, which gives,

since $$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{curl } F = i\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$$

6'. $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

(b) If (6') holds in D and D is simply connected, then the integral is independent of path in D.

Proof:

(a) If the line integral is independent of path in D, then F = grad f by (2) and

$$\text{curl } F = \text{curl (grad } f) = 0 \qquad \text{So, that (6) holds.}$$

(b) The proof of the converse requires "Stokes's theorem" and is omittes here.

Comment For a line integral in the plane

$$\int_C F(r)\cdot dr = \int_C (F_1 dx + F_2 dy),$$

curl F has just one component and (6′) reduces to the single relation 6″.

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

ILLUSTRATIVE EXAMPLES

Example:

Exactness and independence of path. Determination of a potential

Using (6′), show that the differential form under the integral sign of

$$t = \int_C \left[2xyz^2 dx + x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right]$$

is exact, so that we have independence of path in any domain, and find the value of I from A: (0, 0, 1) to B: (1, π/4, 2).

Solution:

Exactness follows from (6″), which gives

$$(F_3)_y = 2x^2z + \cos yz - yz\sin yz = (F_2)_z$$
$$(F_1)_z = 4xyz = (F_3)_x$$
$$(F_2)_x = 2xz^2 = (F_1)_y.$$

To find f, we integrate F_2 (which is "long," so that we save work) and then differentiate to compare with F_1 and F_3.

$$f_x = F_1 = 2xyz^2$$
$$f_y = F_2 = (x^2z^2 + z\cos yz)$$
$$f_z = F_3 = 2x^2yz + y\cos yz)$$
$$f = \int F_2 dy = \int (x^2z^2 + z\cos yz) = x^2z^2y + \sin yz + g(x,z)$$
$$f_x = 2xz^2y + g_x = f_1 = 2xyz^2,\quad g_x = 0 \qquad g = h(z)$$
$$f_z = 2x^2zy + y\cos yz + h' = F_3 = 2x^2zy + y\cos yz,\ h' = 0$$

so that, taking h = 0, we have

$$f(x, y, z) = x^2yz^2 + \sin yz.$$

From this and (3) we get, $I = f(B) - f(A)$

$$= f(1, \pi/4, 2) - f(0, 0, 1) = \pi + \sin\frac{1}{2}\pi - 0 = \pi + 1$$

The assumption in Theorem 3 that D be simply connected is essential and cannot be omitted.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.180 The line integral $\int \vec{V}.d\vec{r}$ of the vector $\vec{V}.(\vec{r}) = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$ from the origin to the point P(1, 1, 1)

(a) is 1
(b) is zero
(c) is −1
(d) cannot be determined without specifying path

[ME, GATE-2005, 2 marks]

Solution: (a)

$$f_x = 2xyz,\ f_y = x^2z,\ f_z = x^2y$$

By integraning, we get f = Potential function of $\vec{V} = x^2yz$

∴ line integral of the vector function from point A (0, 0, 0) to the point B(1, 1, 1) is

$$= f(B) - f(A) = (x^2yz)_{1,1,1} - (x^2yz)_{(0,0,0)} = 1 - 0 = 1$$

Q.181 Given a vector field $\vec{F} = y^2x\hat{a}_x - yz\hat{a}_y - x^2\hat{a}_z$, the line integral $\int \vec{F} \cdot \vec{dl}$ evaluated along a segment on the x-axis from x = 1 to x = 2 is

(a) –2.33 (b) 0

(c) 2.33 (d) 7

[EE, GATE-2013, 1 Mark]

Solution: (b)

To find: $\int \vec{F} \cdot \vec{dl}$ along a segment on the x-axis from x = 1 to x = 2.

i.e. $y = 0, \quad z = 0, \quad dy = 0 \text{ and } dz = 0$

$$\int \vec{F} \cdot \vec{dl} = \int (y^2 x\hat{a}_x - yz\hat{a}_y - x^2\hat{a}_z) \cdot (\hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz)$$

$$= \int y^2 x dx - yz dy - x^2 dz$$

Putting, $y = 0$, $z = 0$, $dy = 0$ and $dz = 0$

We get,

$$\int \vec{F} \cdot \vec{dl} = 0$$

Q.182 Consider points P and Q in the x-y plane, with P = (1,0) and Q = (0,1). The line integral $2\int_P^Q (xdx + ydy)$ along the semicircle with the line segment PQ as its diameter

(a) is –1

(b) is 0

(c) is 1

(d) depends on the direction (clockwise or anti-clockwise) of the semicircle

[EC, GATE-2008, 2 marks]

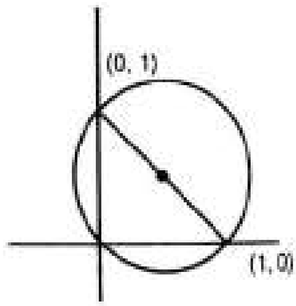
Solution: (b)

Taking f(x, y) = xy, we can show that, xdx + ydy, is exact. So, the value of the integral is independent of path

$$= 2\int_P^Q (x\,dx + y\,dx)$$

$$= 2\int_1^0 x\,dx + 2\int_0^1 y\,dy$$

$$= 2\left[\frac{x^2}{2}\Bigg|_1^0 + \frac{y^2}{2}\Bigg|_0^1\right] = 0$$



or Integral $= f(Q) - f(P) = [xy]_{(0,\,1)} - [xy]_{(1,\,0)} = 0 - 0 = 0$

Q.183 The value of the line integral

$$\int_c (2xy^2dx + 2x^2ydy + dz)$$

along a path joining the origin (0, 0, 0) and the point (1, 1, 1) is

(a) 0 (b) 2

(c) 4 (d) 6

[EE, 2016 : 1 Mark, Set-2]

Solution: (b)

$$\int_C \bar{F}\cdot\bar{dr}$$

where,
$$\bar{F} = xy^2\,\bar{i} + 2x^2y\,\bar{j} + \bar{k}$$
$$\nabla\times\vec{F} = \vec{O}$$

($\vec{F}$ is irrotational $\Rightarrow$ $\vec{F}$ is conservative)

$$\vec{F} = \nabla\phi \qquad (\phi \text{ is scalar potational function})$$
$$\phi_x = 2xy^2$$
$$\phi_y = 2x^2y$$
$$\phi_z = 1$$
$\Rightarrow$
$$\phi = x^2y^2 + z + C$$

where, $\vec{F}$ is conservative

$$\int_C \bar{F}.\bar{dr} = \int_{(0,0,0)}^{(1,1,1)} d\phi = \left[x^2y^2 + z\right]_{(0,0,0)}^{(1,1,1)} = 2$$

Q.184 The line integral of the vector field $F = 5xz\hat{i} + (3x^2 + 2y)\hat{j} + x^2z\hat{k}$ along a path from (0, 0, 0) to (1, 1, 1) parameterized by (t, t^2, t) is ______.

[EE, 2016 : 2 Marks, Set-2]

Solution:

$$E = 5xz\bar{i} + (3x^2 + 2y)\bar{j} + x^2z\bar{k}$$
$$= \int_C \bar{F}.\bar{dr} = \int_C 5xz\,dx + (3x^2 + 2y)dy + x^2z dz$$
$$x = t,\quad y = t^2,\quad z = t,\ t = 0 \text{ to } 1$$
$$dx = dt$$
$$dy = 2t\,dy,\quad dz = dt$$
$$= \int_0^1 5t^2dt + (3t^2 + 2t^2)2t\,dt + t^3dt = \int_0^1 (5t^2 + 11t^3)\,dt$$
$$= \left[\frac{5t^3}{3} + \frac{11t^4}{4}\right]_0^1 = \frac{5}{3} + \frac{11}{4} = \frac{53}{12} = 4.41$$

2.14.21 Green's Theorem in the Plane

Double integrals over a plane region may be transformed into line integrals over the boundary of the region and conversely. This is of practical interest because it may help to make the evaluation of an integral easier. It also helps in the theory whenever one wants to switch from one kind of integral to the other. The transformation can be done by the following theorem.

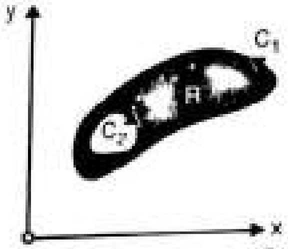
Theorem. 1 (Green's Theorem in The Plane)

(Transformation between double integrals and line integrals)

Let R be a closed bounded region (see Sec. 9.3) in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R. Then.

1. $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy)$

here we integrate along the entire boundary C or R such that R is on the left as we advance in the direction of integration (See Fig. below)



Region R whose boundary is C consists of two parts: C_1 is traversed counterclockwise, while C_2 is traversed clockwise, so that R is on left as we advance.

Comment. Formula (1) can be written in vectorial form

1'. $\iint_R (\text{curl}F) \cdot \hat{k} dxdy = \oint_C F \cdot dr$ $\quad (F = [F_1, F_2] = F_1 i + F_2 j)$

This follows from the fact that the third component of curl F is $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

ILLUSTRATIVE EXAMPLES

Example:

Verification of Green's theorem in the plane.

Green's theorem in the plane will be quite important in our further work. Before proving it, let us get used to it by verifying it for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$

Solution:

In (1) onthe left we get

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint_R [(2y+2) - (2y-7)] dxdy = 9 \iint_R dx\,dy$$
$$= 9\pi$$

(since the circular disk R has area π).

On the right in (1) we represent C(oriented counterclockwise!) by

$$r(t) = [\cos t, \sin t]$$

Then $$r'(t) = [-\sin t, \cos t].$$

On C we thus obtain

$$F_1 = \sin^2 t - 7 \sin t,$$
$$F_2 = 2 \cos t \sin t + 2 \cos t.$$

Hence the integral in (1) on the right becomes

$$\int_C (F_1 x' + F_2 y') dt = \int_C^{2\pi} [(\sin^2 t - 7\sin t)(-\sin t) + 2(\cos t \sin t + \cos t)(\cos t)] dt$$
$$= 0 + 7\pi + 0 + 2\pi = 9\pi.$$

This verifies Green's theorem in the plane.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.185 Value of the integral $\oint_c (xydy - y^2dx)$, where, c is the square cut from the first quadrant by the lines x = 1 and y = 1 will be (Use Green's theorem to change the line integral into double integral)

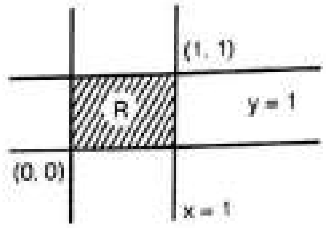
(a) $\frac{1}{2}$ (b) 1

(c) $\frac{3}{2}$ (d) $\frac{5}{3}$ **[CE, GATE-2005, 2 marks]**

Solution: (c)

Green's Theorem is

$$\oint_c \phi dx + \psi dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$$



Here $$I = \oint_c (xydy - y^2dx)$$

$$= \oint_c (-y^2)dx + (xy)dy$$

$\therefore$ $$\phi = -y^2, \psi = xy$$

$$\frac{\partial \psi}{\partial x} = y, \frac{\partial \phi}{\partial y} = -2y$$

Substituting in Green's theorem, we get,

$$I = \int_{y=0}^{1}\int_{x=0}^{1} [y-(-2y)]dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1} 3y\,dx\,dy$$

$$= \int_{y=0}^{1} [3xy]_{x=0}^{1}\,dy = \int_{y=0}^{1} 3ydy = \frac{3}{2}$$

Q.186 The following surface integral is to be evaluated over a sphere for the given steady velocity vector field F = xi + yj + zk defined with respect to a Cartesian coordinate system having i, j and k as unit base vectors.

$$\iint_S \frac{1}{4}(F.n)dA$$

where S is the sphere, $x^2 + y^2 + z^3 = 1$ and n is the outward unit normal vector to the sphere. The value of the surface integral is

(a) π (b) 2π

(c) $3\pi/4$ (d) 4π

[ME, GATE-2013, 2 Marks]

Answer: (a)

Q.187 The value of $\int_C [(3x - 8y^2)dx + (4y - 6xy)dy]$, (where C is the boundary of the region boundary by $x = 0$, $y = 0$ and $x + y = 1$) is ________. **[ME, GATE-2015 : 2 Marks, Set-3]**

Solution: (1.666)

$\int_C [(3x-8y^2)dx + (4y-6xy)dy]$, C is

boundary of region bounded by $x = 0$, $y = 1$, and $z + y = 1$.
Using Green's theorem

$$I = \oint_C (Pdx + Qdy) = \oint_R \oint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Here,
$$P = 3x - 8y^2$$
$$Q = 4y - 6xy$$
$$\frac{\partial Q}{\partial x} = -6y$$
$$\frac{\partial P}{\partial y} = -16y$$
$$I = \iint(-6y-(-16y))dxdy = \iint 10y\,dxdy$$
$$I = 10\int_0^1 dx \int_0^{1-x} \frac{y^2}{2} = 5\int_0^1 dx(1-x)^2$$
$$I = 5\int_0^1 (1-x)^2 .dx = 1.6666$$

Q.188 The value of the line integral $\oint_C \bar{F}\cdot\bar{r}'ds$, where C is a circle of radius $\frac{4}{\sqrt{\pi}}$ units is______

Here, $\bar{F}(x,y) = y\hat{i} + 2x\hat{j}$ and $\bar{r}'$ is the UNIT tangent vector on the curve C at an arc length s from a reference point on the curve $\hat{i}$ and $\hat{j}$ are the basis vectors in the x-y Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction.

[ME, 2016 : 2 Marks, Set-3]

Solution:

$$\int_C \bar{F}\cdot\bar{r}'dx = \int_C \bar{F}\cdot d\bar{r} = \int_C F_1dx + F_2dy$$
$$= \iint_R \left(\frac{\partial F_2}{dx} - \frac{\partial F_1}{dy}\right)dxdy \qquad F_1 = y \quad F_2 = 2x$$
$$= \iint_R (2-1)dxdy \qquad \frac{\partial F_1}{\partial y} = 1 \quad \frac{\partial F_2}{\partial x} = 2$$
$$= \iint dxdy$$
$$= \text{Area of the circle with radius } \frac{4}{\sqrt{\pi}} = \pi\left(\frac{4}{\sqrt{\pi}}\right)^2$$
$$= \pi\frac{16}{\pi} = 16$$

Q.189 A scalar potential φ has the following gradient : $\nabla\varphi = yz\hat{i} + xz\hat{j} + xy\hat{k}$. Consider the integral $\int_C \nabla\varphi . d\vec{r}$ on the curve $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The curve C is parameterized as follows :

$$\begin{cases} x = t \\ y = t \\ z = 3t^2 \end{cases} \text{ and } 1 \le t \le 3$$

The value of the integral is____

[ME, 2016 : 2 Marks, Set-2]

Solution:

$$\int_C \nabla\phi \cdot d\bar{r} = \int_C (yz\bar{i} + xz\bar{p} + xy\bar{k}) \times$$

$$= \int_C yzdx + xzdy + xydz = \int_C d(xyz) = (xyz)$$

Given that $x = t$, $y = t^2$, $z = 3t^2$

$$= (t \cdot t^2 \cdot 3t^2)\Big|_1^3 = 3(t^5)\Big|_1^3$$

$$= 3(3^5 - 1) = 3^6 - 3$$

$$= 729 - 3 = \mathbf{726}$$

2.14.22 Triple Integrals : Divergence Theorem of Gauss

In this section we first discuss triple integrals. Then we obtain the first "big" integral theorem, which transforms surface integrals into triple integrals. It is called **Gauss's divergence theorem** because it involves the divergence of a vector function.

The triple integral is a generalization of the double integral. For defining this integral we consider a function f(x, y, z) defined in a bounded closed region T in space. We subdivide this three-dimensional region T by planes parallel to the three coordinate planes. Then those boxes of subdivision (rectangular parallelopiped) that lie entirely inside T are numbered 1 to n. In each such box we choose an arbitrary point, Say, (x_k, y_k, z_k) in box k, and form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k$$

where DV_k is the volume of box k. This we do for larger and larger positive integers n arbitrarily but so that the maximum length of all the edges of those n boxes approaches zero as n approaches infinity. This gives a sequence of real numbers J_{n_1}, J_{n_2} ,.... We assume that f(x, y, z) is continuous in a domain containing T and T is bounded by finitely many smooth surfaces (see Sec. 9.5). Then it can be shown (See Ref. [5] in Appendix 1) that the sequence converges to a limit that is independent of the choice of subdivisions and corresponding points (x_k, y_k, z_k). This limit is called the triple integral of f(x, y, z) over the region T and is denoted by

$$\iiint_T f(x, y, z)dx\,dy\,dz \quad \text{or} \quad \iiint_T f(x, y, z)dV$$

Triple integrals can be evaluated by three successive integrations. This is similar to the evaluation of double integrals by two successive integrations.

2.14.22.1 Divergence Theorem of Gauss

Triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely. This is of practical interest because one of the two kinds of integral is often simpler than the other. It also helps in establishing fundamental equations in fluid flow, heat conduction, etc., as we shall see. The transformation is done by the divergence theorem, which involves the divergence of a vector function $F = [F_1, F_2, F_3] = F_1i + F_2j + F_3k$,

1. $$\text{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{(Sec. 8.10)}$$

Theorem. 1 (Divergence Theorem of Gauss)

Transformation between volume integrals and surface integrals

Let T be a closed[11] bounded region in space whose boundary is a piecewise smooth orientable surface S. Let F(x, y, z) be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then,

2. $$\iiint_T \text{div}\, F\, dV = \iint_S F \cdot n\, dA$$

where n is the outer unit normal vector of S (pointing to the outside of S, as in Fig. 231).

Formula (2) in Components, using (1) and $n = [\cos\alpha, \cos\beta, \cos\gamma]$, we can write (2)

3*. $$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dA$$

since, $$\iint_S F.n dA = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

equation 2 may also be written as,

3. $$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1\, dy\, dz + F_2 dz\, dx + F_3 dx\, dy)$$

ILLUSTRATIVE EXAMPLES

Example:

Evaluation of a surface integral by the divergence theorem

Before we prove the divergence theorem, let us show a typical application. By transforming to a triple integral, evaluate

$$I = \iint_S (x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy).$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$.

Solution:

In (3) we now have

$$F_1 = x^3, F_2 = x^2 y \cdot F_3 = x^2 z$$

Hence, $\text{div } F = 3x^2 + x^2 + x^2 = 5x^2$

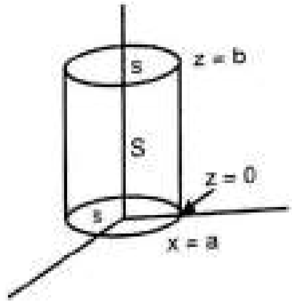
Introducing polar coordinates r, θ defined by x = r cos θ, y = r sin θ (thus, cylindrical coordinates r, θ, z), we have dx dy dz = r dr dθ dz, and we obtain

$$I = \iiint_T 5x^2 dx\, dy\, dz$$

$$= 5\int_{z=0}^{b}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos^2\theta\, r\, dr\, d\theta\, dz$$

$$= 5b\int_0^a\int_0^{2\pi} r^3\cos^2\theta\, dr\, d\theta$$

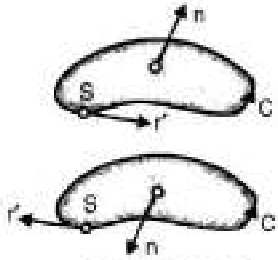
$$= 5b\frac{a^4}{4}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{5}{4}\pi ba^4$$



2.14.22.2 Stokes's Theorem

Having seen the great usefulness of Gauss's theorem, we now turn to the second "big" theorem in this chapter, Stokes's theorem, which transforms line integrals into surface integrals and conversely. Hence this theorem generalizes Green's theorem. It involves the curl,

1. $$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



Stoke's theorem

Theorem. 2 (Stokes' Theorem)

Transformation between surface integrals and line integrals

Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. Let F(x, y, z) be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

2. $$\iint_S (\text{curl } F)\cdot n\, dA = \oint_C F\cdot r'(s)\, ds$$

where n is a unit normal vector of S and, depending on n, the integration around C is taken in the sense shown in Figure above. Furthermore, r' = dr/ds is the unit tangent vector and s the arc length of C. Formula 2 can be written in terms of components:

3. $$\iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right] du\, dv$$

$$= \oint_{\bar{C}} (F_1 dx + F_2 dy + F_3 dz)$$

where R is the region with boundary curve $\bar{C}$ in the uv-plane corresponding to S represented by r(u, v), and $\mathbf{N} = [N_1, N_2, N_3] = \mathbf{r}_u \times \mathbf{r}_v$.

ILLUSTRATIVE EXAMPLES

Example: 1

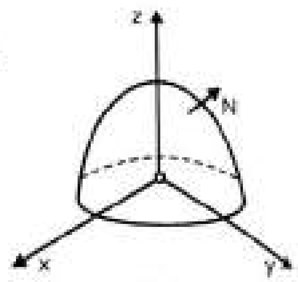
Verification of Stokes's theorem

Before proving Stokes's theorem, let us get used to it by verifying it for F = [y, z, x] yi + zj + xk and S the paraboloid.

$$z = f(x, y) = 1 - (x^2 + y^2),\ z \geq 0.$$

Solution:

The curve C is the circle **r**(s) = [cos s, sin s, 0] = cos s i + sin sj. It has the unit tangent vector **r**′(s) = [–sin s, cos s, 0] = – sin sl + cos **j**. Consequently, the line integral in (2) on the right is simply



Surface S in Example 1

$$\oint_C F \cdot dr = \int_0^{2\pi} [(\sin s)(-\sin s) + 0 + 0] ds = -\pi$$

On the other hand, in (2) on the left we need (verify this)

$$\text{curl } F = [-1, -1, -1]$$

and

$$N = \text{grad}\,(z - f(x, y)) = [2x, 2y, 1]$$

so that (curl F).N = –2x – 2y – 1. From (3) in previous section we get

$$\iint_S (\text{curl}F) \cdot n\,dA = \iint_R (-2x - 2y - 1) dx\,dy$$

$$= \iint_R (-2r\cos\theta - 2r\sin\theta - 1)\, r\, dr\, d\theta$$

where x = r cosθ, y = r sin θ, and dx dy = r dr dθ. Now the projection R of S in the xy-plane is given in polar coordinates by R : r ≤ 1, 0 ≤ θ 2π. The integration of the cosine and sine terms over θ from 0 to 2π gives zero. The remaining term –1. r has integral (– 1/2) 2π = – π, in agreement with the previous result. Note well that N is an upper normal vector of S, and r(s) orients C counterclockwise, as required in Stokes's theorem.

Example: 2

Green's theorem in the plane as a special case of Stokes's theorem

Let $F = [F_1, F_2] = F_1 i + F_2 j$ be a vector function that is continuously differentiable in a domain in the xy-plane containing a simply connected bounded closed region S whose boundary C is a piecewise smooth simple closed curve. Then, according to (1),

$$(\text{curl } F)\cdot a = (\text{curl } F)\cdot k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Solution:

Hence the formula in stoke's theorem now takes the form

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C (F_1 dx + F_2 dy)$$

This shows that Green's theorem in the plane is a special case of Stokes's theorem.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.191 Stokes theorem connects

(a) a line integral and a surface integral
(b) a surface integral and a volume integral
(c) a line integral and a volume integral
(d) gradient of a function and its surface integral

[ME, GATE-2005, 1 mark]

Solution: (a)

A line integral and a surface integral is related by stroke's theorem.

Q.192 Which one of the following describes the relationship among the three vectors, $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+4\hat{k}$?

(a) The vectors are mutually perpendicular
(b) The vectors are linearly dependent
(c) The vectors are linearly independent
(d) The vectors are unit vectors

[ME, 2014 : 1 Mark, Set-1]

Solution : (b)

For linear dependency, det $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 6 & 4 \end{vmatrix}$ must be zero.

$\therefore \quad \Delta = 1(12-6) - 1(8-5) + 1(12-15) = 6-3-3 = 0$

$\therefore$ There three vectors are linearly dependent.

Q.193 Curl of vector $\vec{F} = x^2z^2\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$ is

(a) $(4yz^3+2xy^2)\hat{i}+2x^2z\hat{j}-2y^2z\hat{k}$
(b) $(4yz^3+2xy^2)\hat{i}-2x^2z\hat{j}-2y^2z\hat{k}$
(c) $2xz^2\hat{i}-4xyz\hat{j}+6y^2z^2\hat{k}$
(d) $2xz^2\hat{i}+4xyz\hat{j}+6y^2z^2\hat{k}$

[ME, 2014 : 1 Mark, Set-2]

Solution : (a)

$$\vec{F} = x^2z^2\vec{i} - 2xy^2z\vec{j} + 2y^2z^3\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & -2xy^2z^2 & 2y^2z^3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2y^2z^3)+\frac{\partial}{\partial z}(2xy^2z)\right] - \vec{j}\left[\frac{\partial}{\partial y}(2y^2z^3)-\frac{\partial}{\partial z}(x^2z^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-2xy^2z)-\frac{\partial}{\partial y}(x^2z^2)\right]$$

$$\nabla\times\vec{F} = \vec{i}[4yz^3+2xy^2] - \vec{j}[2zx^2] + \vec{k}[-2y^2z-0]$$

$$= (4yz^3+2xy^2)\vec{i} - (2x^2z)\vec{j} - (2y^2z)\vec{k}$$

Q.194 Divergence of the vector field $x^2z\hat{i}+xy\hat{j}-yz^2\hat{k}$ at $(1, -1, 1)$ is

(a) 0
(b) 3
(c) 5
(d) 6

[ME, 2014 : 1 Mark, Set-3]

Solution : (c)

$$\vec{F} = x^2 z\hat{i} + xy\hat{j} - yz^2\hat{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(yz^2)$$

$$\nabla \cdot \vec{F} = 2xz + x - 2yz$$

$$\therefore \quad \nabla \cdot \vec{F}\Big|_{(1,-1,1)} = 2 \times 1 \times 1 + 1 - 2 \times -1 \times 1 = 2 + 1 + 2 = 5$$

Q.195 The surface integral $\iint_S \frac{1}{\pi}(9xi - 3yj) \cdot n\,dS$ over the sphere given by $x^2 + y^2 + z^2 = 9$ is ____

[ME, 2015 : 2 Marks, Set-2]

Solution: (216)

Accodring to gauge divergence theorem

$$\iint_S \frac{1}{\pi}(9xi - 3yj) \cdot n\,dS = \frac{1}{\pi}\int \text{divergence } (9xi - 3yj).dv$$

$$= \frac{1}{\pi}[9-3] \times \frac{4}{3}\pi\left[r^3\right]$$

$r = 3$ [given]

$$= \frac{1}{\pi} \times 6 \times \frac{4}{3}\pi \times 27 = 216$$

Q.196 Let $\nabla \cdot (f\vec{v}) = x^2y + y^2z + z^2x$, where f and v are scalar and vector fields respectively. If $\vec{v} = y\vec{i} + z\vec{j} + x\vec{k}$, then $\vec{v} \cdot \nabla f$ is

(a) $x^2y + y^2z + z^2x$
(b) $2xy + 2yz + 2zx$
(c) $x + y + z$
(d) 0

[EE, 2014 : 1 Mark, Set-3]

Solution : (a)

$$\vec{v} = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\hat{i}\frac{\partial(fV)}{\partial x} + \hat{j}\frac{\partial(fV)}{\partial y} + \hat{k}\frac{\partial(fV)}{\partial z} = x^2y + y^2z + z^2x$$

$$y\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial z} = x^2y + y^2z + z^2x \quad ...(i)$$

$$\vec{v}\,\Delta f = (y\hat{i} + z\hat{j} + x\hat{k})\left(\frac{\hat{i}\partial}{\partial x} + \frac{\hat{j}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)f$$

$$\vec{v}\,\Delta f = \frac{y\partial f}{\partial x} + \frac{z\partial f}{\partial y} + \frac{x\partial f}{\partial z} \quad ...(ii)$$

From equations (i) and (ii)

$$\vec{v} \cdot \nabla f = x^2y + y^2z + z^2x$$

Q.197 $\iint(\nabla \times P) \cdot ds$, where P is a vector, is equal to

(a) $\oint P \cdot dl$

(b) $\oint \nabla \times \nabla \times P \cdot dl$

(c) $\oint \nabla \times P \cdot dl$

(d) $\iiint \nabla \cdot P dv$

[EC, GATE-2006, 1 mark]

Solution: (a)

$\iint(\Delta \times P) \cdot ds = \oint P \cdot dl$ (stokes Theorem)

Q.198 Consider a vector field $\vec{A}(\vec{r})$. The closed loop line integral $\oint \vec{A} \cdot \vec{dl}$ can be expressed as

(a) $\oiint(\nabla \times \vec{A}) \cdot \vec{ds}$ over the closed surface bounded by the loop

(b) $\oiiint(\nabla \cdot \vec{A}) dv$ over the closed volume bounded by the top

(c) $\iiint(\nabla \cdot \vec{A}) dv$ over the open volume bounded by the loop

(d) $\iint(\nabla \times \vec{A}) \cdot \vec{ds}$ over the open surface bounded by the loop

[EC, GATE-2013, 1 Mark]

Solution: (d)

According to Stoke's theorem

$$\oint_c \vec{A} \cdot \vec{dl} = \iint_s (\nabla \times \vec{A}) \cdot \vec{ds}$$

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CHAPTER 3

Differential Equations

3.1 INTRODUCTION

Differential equations are fundamental in engineering mathematics since many of the physical laws and relationships between physical quantities appear mathematically in the form of such equations.

The transition from a given physical problem to its mathematical representation is called modeling. This is of great practical interest to engineer, physicist or computer scientist. Very often, mathematical models consist of a differential equations or system of simultaneous differential equations, which needs to be solved. In this chapter we shall look at classifying differential equations and solving them by various standard methods.

3.2 DIFFERENTIAL EQUATIONS OF FIRST ORDER

3.2.1 Definitions

A differential equation is an equation which involves derivatives or differential coefficients or differentials. Thus the following are all examples of differential equations.

(a) $x^2dx + y^2\,dy = 0$

(b) $\frac{d^2x}{dt^2} + a^2x = 0$

(c) $y = x\frac{dy}{dx} + \frac{x^2}{dy/dx}$

(d) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-5/3} = a\frac{d^2y}{dx^2}$

(e) $\frac{dx}{dt} - wy = a\cos pt,\ \frac{dy}{dt} + wx = a\sin pt$

(f) $x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

(g) $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

An **ordinary differential equations** is that in which all the differential coefficients all with respect to a single independent variable. Thus the equations (a) to (d) are all ordinary differential equations. (e) is a **system** of ordinary differential equations.

A **partial differential equations** is that in which there are two or more independent variables and partial differential coefficients with respect to any of them. The equations (f) and (g) are partial differential equations.

The **order** of a differential equation is the order of the highest derivative appearing in it. The **degree** of a differential equation is the degree of the highest derivative occurring in its, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Thus from the examples above,

(a) is of the first order and first degree;

(b) is of the second order and first degree;

(c) written as $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x^2$ is of the first order but of second degree;

(d) After removing radicals is written as $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-5} = a^3\left(\frac{d^2y}{dx^2}\right)^3$ and is of the second order and third degree.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 The degree of the differential equation $\frac{d^2x}{dt^2} + 2x^3 = 0$ is

(a) 0
(b) 1
(c) 2
(d) 3

[CE, GATE-2007, 1 mark]

Solution: (b)

Degree of a differential equation is the power of its highest order derivative after the differential equation is made free of radicals and fractions if any, in derivative power.

Hence, here the degree is 1, which is power of $\frac{d^2x}{dt^2}$

Q.2 The order and degree of the differential equation $\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$ are respectively

(a) 3 and 2
(b) 2 and 3
(c) 3 and 3
(d) 3 and 1

[CE, GATE-2010, 1 mark]

Solution: (a)

$$\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$$

Removing radicals we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = 16\left[\left(\frac{dy}{dx}\right)^3 + y^2\right]$$

$\therefore$ The order is 3 since highest differential is $\frac{d^3y}{dx^3}$

The degree is 2 since power of highest differential is 2.

Q.3 The Blasius equation, $\frac{d^3f}{d\eta^3} + \frac{f}{2}\frac{d^2f}{d\eta^2} = 0$, is a

(a) second order nonlinear ordinary differential equation
(b) third order nonlinear ordinary differential equation
(c) third order linear ordinary differential equation
(d) mixed order nonlinear ordinary differential equation

[ME, GATE-2010, 1 mark]

Solution: (b)

$\frac{d^3f}{dn^3} + \frac{f}{2}\frac{d^2f}{dn^2} = 0$ is third order $\left(\frac{d^3f}{dn^3}\right)$ and it is non linear, since the product $f \times \frac{d^2f}{dn^2}$ is not allowed in linear differential equation.

Q.4 The partial differential equation

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ is a

(a) linear equation of order 2
(b) non-linear equation of order 1
(c) linear equation of order 1
(d) non-linear equation of order 2

[ME, GATE-2013, 1 Mark]

Solution: (d)

In the equation, dependant variable multiplied with derivative, so it is not a linear equation.
$\therefore$ given differential equation is non-linear equation of order '2'.

Q.5 Consider the following differential equation:

$\frac{dy}{dt} = -5y$; initial condition: $y = 2$ at $t = 0$

The value of y at $t = 3$ is

(a) $-5e^{-10}$
(b) $2e^{-10}$
(c) $2e^{-15}$
(d) $-15e^{2}$

[ME, GATE-2015 : 2 Marks, Set-2]

Solution: (c)

$$\frac{dy}{dt} = -5y$$

$$\int \frac{dy}{y} = -\int 5\,dt$$

$$\ln y = -5t + C$$

at $t = 0$

$y = 2$

$$\ln 2 = C$$

So, $\ln y = -5t + \ln 2$

$$\ln\frac{y}{2} = -5t$$

$$\frac{y}{2} = e^{-5t}$$

$$y = 2e^{-5t}$$

at $$t = 3$$

$$y = 2e^{-15}$$

Q.6 The following differential equation has $3\left(\frac{d^2y}{dt^2}\right) + 4\left(\frac{dy}{dt}\right)^3 + y^2 + 2 = x$

(a) degree = 2, order = 1
(b) degree = 1, order = 2
(c) degree = 4, order = 3
(d) degree = 2, order = 3

[EC, GATE-2005, 1 mark]

Solution: (b)

Order is highest derivative term, so order = 2. Degree is power of highest derivative term. So, degree = 1.

Q.7 The order of the differential equation $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is

(a) 1
(b) 2
(c) 3
(d) 4

[EC, GATE-2009, 1 mark]

Solution: (b)

Highest derivative of differential equation is 2.

3.2.2 Solution of a Differential Equation

A solution (or integral) of a differential equation is a relation between the variable which satisfies the given differential equation.

For example, $$y = ce^{\frac{x^3}{3}} \quad \text{... (i)}$$

is a solution of $$\frac{dy}{dx} = x^2 y \quad \text{... (ii)}$$

The **general** (or **complete**) **solution** of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (i) is a general solution of (ii) as the number of arbitrary constants (one constant c) is the same as the order of the equations (ii) (first order). Similarly, in the general solution of a second order differential equation, there will be two arbitrary constants.

A **particular solution** is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example $$y = 4e^{\frac{x^3}{3}}$$

is a particular solution of the equation (ii), as it can be derived from the general solution (i) by putting c = 4.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and usually is not of much practical interest in engineering.

3.2.3 Equations of the First Order and First Degree

It is not possible to analytically solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations:

1. Equations where variables are separable.,
2. Homogenous equations,
3. Linear equations,
4. Exact equations.

In other cases, the particular solution may be determined numerically.

3.2.3.1 Variables Separable

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is $f(y)\,dy = \phi(x)dx$.

Integrating both sides, we get $\int f(y)dy = \int \phi(x)dx + c$ as its solution.

ILLUSTRATIVE EXAMPLES

Example: 1

Solve $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

Solution:

Given equation is $\frac{dy}{dx} = e^y(e^x + x^2)$

or $e^{-y}\,dy = (e^x + x^2)dx$

Integrating both sides, $\int e^{-y}dy = \int (e^x + x^2)dx + c$

$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

$$3e^{-y} = -3e^x - x^3 + c' \qquad [c' = -3c]$$

Note 1: In the above line, we have introduced a new arbitrary constant c′ instead of c, in order to put the result in a better form. Such changes are allowed and often made.

Note 2: Intial value problem: A differential equation together with an initial condition is called an **initial value problem.** It is of the form given in the next example. The condition y(0) = 0 in the example below is called an initial condition. It is used to determine the value of the arbitrary constant in the general solution. In a second order differential equation, two such condions will be required, since there will be two arbitrary constants which will need to be determined.

Example: 2

Solve $dy/dx = (x + y + 1)^2$, if $y(0) = 0$.

Solution:

Putting $x + y + 1 = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$.

∴ The given equation becomes $\frac{dt}{dx} - 1 = t^2$ or $\frac{dt}{dx} = 1 + t^2$

Integrating both sides, we get $\int \frac{dt}{1+t^2} = \int dx + c$

or $\tan^{-1} t = x + c$

or $\tan^{-1}(x + y + 1) = x + c$

or $x + y + 1 = \tan(x + c)$

When $x = 0, y = 0$

$\Rightarrow \quad 1 = \tan(c)$

$\Rightarrow \quad c = \frac{\pi}{4}$

Hence the solution is $x + y + 1 = \tan(x + \pi/4)$.

Note: Equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to the 'variable separable' form by putting $ax + by + c = t$.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.8 Biotransformation of an organic compound having concentration (x) can be modeled using an ordinary differential equation $\frac{dx}{dt} + kx^2 = 0$, where k is the reaction rate constant. If x = a at t = 0, the solution of the equation is

(a) $x = ae^{-kt}$

(b) $\frac{1}{x} = \frac{1}{a} + kt$

(c) $x = a(1 - e^{-kt})$

(d) $x = a + kt$ **[CE, GATE-2004, 2 marks]**

Solution: (b)

$$\frac{dx}{dt} = -kx^2 \qquad \text{(Note: This is in variable separable form)}$$

$$\Rightarrow \quad \frac{dx}{x^2} = -kdt$$

Integrating both sides, $\int \frac{dx}{x^2} = -\int kdt$

$$-\frac{1}{x} = -kt + C$$

$$\Rightarrow \quad \frac{1}{x} = kt + C'$$

at $t = 0, x = a$

$$\Rightarrow \quad \frac{1}{a} = k \times 0 + C'$$

$$\Rightarrow \quad C' = \frac{1}{a}$$

$$\therefore \quad \frac{1}{x} = kt + \frac{1}{a}$$

Q.9 A spherical naphthalene ball exposed to the atmosphere loses volume at a rate proportional to its instantaneous surface area due to evaporation. If the initial diameter of the ball is 2 cm and the diameter reduces to 1 cm after 3 months, the ball completely evaporates in

(a) 6 months

(b) 9 months

(c) 12 months

(d) infinite time

[CE, GATE-2006, 2 marks]

Solution: (a)

$$\frac{dV}{dt} = -kA \quad \text{...(i)}$$

where $V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$

$\Rightarrow \quad \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

Substituting these in (i) we get,

$$4\pi r^2 \frac{dr}{dt} = -k(4\pi r^2)$$

$$\frac{dr}{dt} = -k$$

$\Rightarrow \quad dr = -kdt$

Integrating we get

$$r = -kt + C$$

at $t = 0,\ r = 1$

$\Rightarrow \quad 1 = -k \times 0 + C$

$\Rightarrow \quad C = 1$

$\therefore \quad r = -kt + 1 \quad \text{...(ii)}$

Now at t = 3 months $\quad r = 0.5$ cm

$\therefore \quad 0.5 = -k \times 3 + 1$

$\Rightarrow \quad k = \frac{0.5}{3}$

Now substituting this value of R in equation (ii) we get,

$$r = -\frac{0.5}{3}t + 1$$

putting $r = 0$ (ball completely evaporates)

in above and solving for t gives $0 = -\frac{0.5}{3}t + 1$

$\Rightarrow \quad t = 6$ months

Q.10 The solution for the differential equation $\frac{dy}{dx} = x^2y$ with the condition that y = 1 at x = 0 is

(a) $y = e^{\frac{1}{2x}}$

(b) $\ln(y) = \frac{x^3}{3} + 4$

(c) $\ln(y) = \frac{x^2}{2}$

(d) $y = e^{\frac{x^3}{3}}$

[CE, GATE-2007, 1 mark]

Solution: (d)

$$\frac{dy}{dx} = x^2y$$

This is variable separable form

$$\frac{dy}{y} = x^2dx$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\Rightarrow \quad \log_e y = \frac{x^3}{3} + C_1$$

$$\Rightarrow \quad y = e^{\frac{x^3}{3} + C_1} = e^{C_1} \times e^{\frac{x^3}{3}}$$

$$y = C \times e^{\frac{x^3}{3}}$$

Now at x = 0, y = 1

$$1 = C \times e^{\frac{0}{3}}$$

$$\Rightarrow \quad C = 1$$

$$\therefore \quad y = e^{\frac{x^3}{3}} \text{ is the solution}$$

Q.11 Solution of $\frac{dy}{dx} = -\frac{x}{y}$ at x = 1 and y = $\sqrt{3}$ is

(a) $x - y^2 = -2$
(b) $x + y^2 = 4$
(c) $x^2 - y^2 = -2$
(d) $x^2 + y^2 = 4$

[CE, GATE-2008, 2 marks]

Solution: (d)

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow \quad y\,dy = -x\,dx$$

$$\Rightarrow \quad \int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

at $x = 1, y = \sqrt{3}$

$$\therefore \quad \frac{(\sqrt{3})^2}{2} = \frac{-1^2}{2} + C$$

$$\Rightarrow \quad C = 2$$

$\therefore$ Solution is $\frac{y^2}{2} = \frac{-x^2}{2} + 2$

$$\Rightarrow \quad x^2 + y^2 = 4$$

Q.12 Solution of the differential equation $3y\frac{dy}{dx} + 2x = 0$ represents a family of

(a) ellipses
(b) circles
(c) parabolas
(d) hyperbolas

[CE, GATE-2009, 2 marks]

Solution: (a)

$$3y\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{3y}$$

$$3y\,dy = -2x\,dx$$

$$\Rightarrow \int 3y\,dy = \int -2x\,dx$$

$$\Rightarrow \frac{3}{2}y^2 = -2\times\frac{x^2}{2} + C$$

$$\Rightarrow 3y^2 + 2x^2 = C$$

$$\Rightarrow \frac{x^2}{\left(\frac{1}{2}\right)} + \frac{y^2}{\left(\frac{1}{3}\right)} = C$$

$$\Rightarrow \frac{x^2}{\left(\frac{1}{2}C\right)} + \frac{y^2}{\left(\frac{1}{3}C\right)} = 1$$

which is the equation of a family of ellipses.

Q.13 The solution of the ordinary differential equation $\frac{dy}{dx} + 2y = 0$ for the boundary condition, $y = 5$ at $x = 1$ is

(a) $y = e^{-2x}$
(b) $y = 2e^{-2x}$
(c) $y = 10.95\,e^{-2x}$
(d) $y = 36.95\,e^{-2x}$

[CE, GATE-2012, 2 mark]

Solution: (d)

Given, $\frac{dy}{dx} + 2y = 0$ and $y(1) = 5$

$$\frac{dy}{dx} = -2y$$

$$\int \frac{dy}{y} = \int -2dx$$

$$\Rightarrow \ln y = -2x + c$$

$$\Rightarrow y = e^{-2x}\cdot e^{c} = c_1 e^{-2x}$$

$$y(1) = c_1 e^{-2} = 5 \quad \Rightarrow \quad c_1 = \frac{5}{e^{-2}}$$

So,
$$y = \frac{5}{e^{-2}}e^{-2x} = 5e^2 e^{-2x}$$
$$= 36.95\,e^{-2x}$$

Q.14 Consider the following difference equation

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

Which of the following is the solution of the above equation (c is an arbitrary constant)?

(a) $\frac{x}{y}\cos\frac{y}{x} = c$
(b) $\frac{x}{y}\sin\frac{y}{x} = c$
(c) $xy\cos\frac{y}{x} = c$
(d) $xy\sin\frac{y}{x} = c$

[CE, GATE-2015 : 2 Marks, Set-I]

Solution: (c)

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

$$\frac{ydx + xdy}{xdy - ydx} = \frac{y}{x}\tan\frac{y}{x}$$

Let

$$y = v \cdot x$$

$$dy = vdx + xdv$$

$$\frac{vxdx + vxdx + x^2dv}{vxdx + x^2dv - vxdx} = v\tan v$$

$$\frac{xdv + 2vdx}{xdv} = v\tan v$$

$$1 + \frac{2v}{x}\frac{dx}{dv} = v\tan v$$

$$\frac{2v}{x}\frac{dx}{dv} = v\tan v - 1$$

$$2\frac{dx}{x} = \left(\tan v - \frac{1}{v}\right)dv$$

Integrating both sides.

$2\log x = \log|\sec v| - \log v + \log c$

$\Rightarrow$ $$x^2 = \frac{c\sec v}{v}$$

$\Rightarrow$ $$x^2\frac{y}{x} = c\sec\frac{y}{x}$$

$\Rightarrow$ $$xy\cos\frac{y}{x} = c$$

Q.15 Consider the following second order linear differential equation

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

The boundary conditions are: at $x = 0$, $y = 5$ and $x = 2$, $y = 21$
The value of y at $x = 1$ is ________.

[CE, GATE-2015 : 2 Marks, Set-II]

Solution:

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

Integrating both sides w.r.t. x

$$\frac{dy}{dx} = -4x^3 + 12x^2 - 20x + c_1$$

Integrating both sides w.r.t. x

$$y = -x^4 + 4x^3 - 10x^2 + c_1x + c_2 \quad ...(i)$$

At $x = 0$, $y = 5$

$\Rightarrow$ $$5 = c_2$$

At $x = 2$, $y = 21$

$\Rightarrow$ $$21 = -16 + 32 - 40 + 2c_1 + c_2$$

$$2c_1 = 21 + 16 - 2 + 40 - 5$$
$$2c_1 = 40$$
$$c_1 = 20$$
$$\Rightarrow \quad y = -x^4 + 4x^3 - 10x^2 + 20x + 5$$

Put $x = 1$

$$\Rightarrow \quad y = -1 + 4 - 10 + 20 + 5 = 18$$

Q.16 The solution of the differential equation $\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x+c}$
(b) $y = \frac{-x^3}{3} + c$
(c) ce^x
(d) unsolvable as equation is non-linear

[ME, GATE-2003, 2 marks]

Solution: (a)

Given diffenential equation

$$\frac{dy}{dx} + y^2 = 0$$
$$\Rightarrow \quad -\frac{dy}{y^2} = dx$$

On integrating, we get $-\int \frac{dy}{y^2} = \int dx$

$$\frac{1}{y} = x + c$$
$$\therefore \quad y = \frac{1}{x+c}$$

Q.17 The solution of dy/dx = y^2 with initial value y (0) = 1 bounded in the interval

(a) $-\infty \le x \le \infty$
(b) $-\infty \le x \le 1$
(c) $x < 1, x > 1$
(d) $-2 \le x \le 2$

[ME, GATE-2007, 2 marks]

Solution: (c)

Given $\frac{dy}{dx} = y^2$

$$\Rightarrow \quad \int \frac{dy}{y^2} = \int dx$$
$$\Rightarrow \quad -\frac{1}{y} = x + c$$
$$\therefore \quad y = -\frac{1}{x+c}$$

When $x = 0$

$y = 1$

$\therefore \quad C = -1$

$$\therefore \quad y = -\frac{1}{x-1}$$

y is bounded when $x - 1 \neq 0$

i.e. $x \neq 1$

i.e. $x < 1$ or $x > 1$

Q.18 Consider the differential equation $\frac{dy}{dx} = \left(1+y^2\right)x$. The general solution with constant c is

(a) $y = \tan\frac{x^2}{2} + \tan c$

(b) $y = \tan^2\left(\frac{x}{2}+c\right)$

(c) $y = \tan^2\left(\frac{x}{2}\right)+c$

(d) $y = \tan\left(\frac{x^2}{2}+c\right)$

[ME, GATE-2011, 2 mark]

Solution: (d)

$$\frac{dy}{dx} = (1+y^2)x$$

$$\int\frac{dy}{1+y^2} = \int x\,dx$$

$$\tan^{-1} y = \frac{x^2}{2}+C$$

$$y = \tan\left(\frac{x^2}{2}+C\right)$$

Q.19 The solution of the first order differential equation $x'(t) = -3x(t)$, $x(0) = x_0$ is

(a) $x(t) = x_0 e^{-3t}$

(b) $x(t) = x_0 e^{-3}$

(c) $x(t) = x_0 e^{-1/3}$

(d) $x(t) = x_0 e^{-1}$

[EE, GATE-2005, 1 mark]

Solution: (a)

Given, $\dot{x}(t) = -3\,x(t)$

i.e. $\frac{dx}{dt} = -3\,x$

$$\frac{dx}{x} = -3\,dt$$

$$\int\frac{dx}{x} = \int -3\,dt$$

$\Rightarrow$ $\ln x = -3t + C$

$\Rightarrow$ $x = e^{-3t+C} = e^C \times e^{-3t}$

putting $e^C = C_1$

$x = C_1 \times e^{-3t}$

Now putting initial condition $x(0) = x_0$

$x_0 = C_1 e^0 = C_1$

$\therefore$ $C_1 = x_0$

$\therefore$ Solution is $x = x_0 e^{-3t}$

i.e. $x(t) = x_0 e^{-3t}$

Q.20 With K as a constant the possible for the first order differential equation $\frac{dy}{dx} = e^{-3x}$ is

(a) $-\frac{1}{3}e^{-3x} + K$
(b) $-\frac{1}{3}e^{3x} + K$
(c) $-3e^{-3x} + K$
(d) $-3e^{-x} + K$

[EE, GATE-2011, 1 mark]

Solution: (a)

$$\frac{dy}{dx} = e^{-3x}$$

$$\int dy = \int e^{-3x}dx$$

$$y = \frac{e^{-3x}}{-3} + K = -\frac{1}{3}e^{-3x} + K$$

Q.21 A differential equation $\frac{di}{dt} - 0.2i = 0$ is applicable over $-10 < t < 10$. If $i(4) = 10$, then $i(-5)$ is _____.

[EE, GATE-2015 : 2 Marks, Set-2]

Solution: (1.652)

$$\frac{di}{dt} = 0.2i$$

$$\frac{di}{i} = 0.2\,dt$$

$$\int \frac{di}{i} = \int 0.2\,dt$$

$$\log i = 0.2t + \log C$$

$$\log i - \log C = 0.2t$$

$$\log\left(\frac{i}{C}\right) = 0.2t$$

$$\frac{i}{C} = e^{0.2t}$$

$$i = Ce^{0.2t} \quad \text{...(i)}$$

$i(4) = 10$ i.e. $i = 10$ when $t = 4$

$$10 = Ce^{(0.2)4}$$

$$10 = C(2.225)$$

$\therefore$ $$C = 4.493 \quad \text{...(ii)}$$

$$i = (4.493)\,e^{0.2t}$$

when, $$t = -5$$

$$i = (4.493)\,e^{(0.2)(-5)} = 1.652$$

Q.22 Which of the following is a solution to the differential equation $\frac{dx(t)}{dt} + 3x(t) = 0$?

(a) $x(t) = 3e^{-t}$

(b) $x(t) = 2e^{-3t}$

(c) $x(t) = -\frac{3}{2}t^2$

(d) $x(t) = 3t^2$

Solution: (b) **[EC, GATE-2008, 1 mark]**

$$\frac{dx}{dt} = -3x$$

$$\frac{dx}{x} = -3dt$$

$$\int \frac{dx}{x} = \int -3dt$$

$$\ln x = -3t + c$$

$$\Rightarrow x = e^{-3t+c}$$

$$\Rightarrow x = e^{c} \cdot e^{-3t} = c_1 e^{-3t} \qquad (c_1 = e^c)$$

$$\Rightarrow x = c_1 e^{-3t}$$

Q.23 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I

A. $\frac{dy}{dx} = \frac{y}{x}$

B. $\frac{dy}{dx} = -\frac{y}{x}$

C. $\frac{dy}{dx} = \frac{x}{y}$

D. $\frac{dy}{dx} = -\frac{x}{y}$

List-II

1. Circles
2. Straight lines
3. Hyperbolas

Codes:

	A	B	C	D
(a)	2	3	3	1
(b)	1	3	2	1
(c)	2	1	3	3
(d)	3	2	1	2

[EC, GATE-2009, 2 marks]

Solution: (a)

A.
$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + \log c = \log cx$$

$$y = cx \qquad \text{... Equation of straight line.}$$

B.
$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{y} = \frac{-dx}{x} \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\log y = -\log x + \log c$$
$$\log y + \log x = \log c$$
$$\log yx = \log c$$
$$yx = c$$
$$y = c/x \quad \text{... Equation of hyperbola}$$

C. $$\frac{dy}{dx} = \frac{x}{y},\ y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx$$
$$\frac{y^2}{2} - \frac{x^2}{2} = \frac{c^2}{2} \rightarrow \text{const}$$
$$y^2 - x^2 = c^2$$
$$\frac{y^2}{c^2} - \frac{x^2}{c^2} = 1 \quad \text{... Equation of hyperbola.}$$

D. $$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow \int y\,dy = -\int x\,dx$$
$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c^2}{2}$$
$$\frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$$
$$x^2 + y^2 = c^2 \quad \text{... Equation of a circle}$$

Q.24 The solution of the differential equation $\frac{dy}{dx} = ky$, $y(0) = c$ is

(a) $x = ce^{-ky}$
(b) $x = ke^{cy}$
(c) $y = ce^{kx}$
(d) $y = ce^{-kx}$

[EC, GATE-2011, 1 marks]

Solution: (c)

$$\frac{dy}{dx} = ky$$
$$\Rightarrow \quad \frac{dy}{y} = k\,dx$$

Integrating both sides

$$\ln y = kx + A$$
Put, $$x = 0$$
$$\ln y(0) = A$$
$$\therefore \quad A = \ln c \qquad [\because y(0) = c]$$
Hence,
$$\ln y = kx + \ln c$$
$$\Rightarrow \quad \ln y - \ln c = kx$$
$$\Rightarrow \quad \ln\left(\frac{y}{c}\right) = kx$$
$$\Rightarrow \quad \frac{y}{c} = e^{kx}$$
or
$$y = c\,e^{kx}$$

Q.25 The general solution of the differential equation $\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x}$ is

(a) $\tan y - \cot x = c$ (c is a constant)
(b) $\tan x - \cot y = c$ (c is a constant)
(c) $\tan y + \cot x = c$ (c is a constant)
(d) $\tan x + \cot y = c$ (c is a constant)

[EC, GATE-2015 : 1 Mark, Set-2]

Solution: (c)

$$\frac{dy}{1+\cos 2y} = \frac{dx}{1-\cos 2x}$$

$$\frac{dy}{2\cos^2 y} = \frac{dx}{2\sin^2 x}$$

$$\sec^2 y\, dy = \operatorname{cosec}^2 x\, dx$$

Integrating both sides, we get

$$\tan y = -\cot x + c$$

$$\tan y + \cot x = c$$

Q.26 The type of the partial differential equation $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ is

(a) Parabolic
(b) Elliptic
(c) Hyperbolic
(d) Non-linear

[IN, GATE-2013 : 1 mark]

Answer: (a)

3.2.3.2 Homogeneous Equations

Homogeneous equations are of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

where f(x, y) and ϕ (x) ϕ(x, y) homogeneous functions of the same degree in x and y.

Homogeneous Function: An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_ny^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as $x^n [a_0 + a_1(y/x) + a_2(y/x)^2 + ...+a_n(y/x)^n]$.

Thus any functions f(x, y) which can be expressed in the form $x^nf(y/x)$, is called a homogeneous function of degree n in x and y. For instance $x^3 \cos(y/x)$ is a homogeneous function of degree 3 in x and y.

To solve a homogeneous equation

1. Put y = vx, then $\frac{dy}{dx} = v + x\frac{dy}{dx}$.
2. Separate the variables v and x, and integrate.

ILLUSTRATIVE EXAMPLES

Example:

Solve $(y^2 - x^2)\,dx - 2xy\,dy = 0$.

Solution:

Given equation is $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ which is homogeneous in x and y. ... (i)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

∴ Eq. (i) becomes $v + x\frac{dv}{dx} = \frac{1}{2}\left[v - \frac{1}{v}\right]$

or $x\frac{dv}{dx} = \frac{1}{2}\left[\frac{v^2-1}{v}\right] - v = \frac{-\left[v^2+1\right]}{2v}$

Separating the variables,

$$\frac{2v}{1+v^2}dv = -\frac{dx}{x}$$

Integrating both sides,

$$\int\frac{2v\,dv}{1+v^2} = -\int\frac{dx}{x} + c$$

or $\ln(1 + v^2) = -\ln x + c = \ln\frac{1}{x} + \ln c_1$

or $\ln(1 + v^2) = \ln\left(\frac{c_1}{x}\right)$

$$1 + v^2 = \frac{c_1}{x}$$

replacing v by $\frac{y}{x}$, we get

$$1+\left(\frac{y}{x}\right)^2 = \frac{c}{x}$$

or $x^2 + y^2 = cx$

or $\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$

This general solution represents a family of circles with centres on the x-axis at $\left(\frac{c}{2}, 0\right)$ and radius $= \frac{c}{2}$, thus passing through origin as shown below.

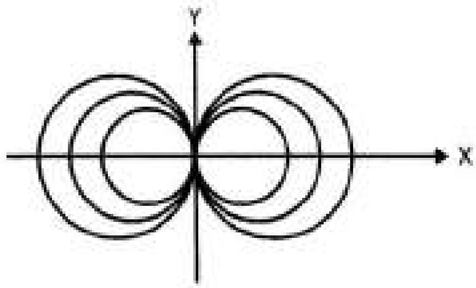

Fig. General Solution (Family of circles)

3.2.3.3 Linear Equations of First Order

A differential equation is said be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus the following differential equations are linear

1. $\frac{dy}{dx} + 4y = 2$
2. $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 4y = 2$

equation (i) is linear first order differential equation while equation (ii) in linear second order differential equation. The following equations are not linear

1. $\left(\frac{dy}{dx}\right)^2 + y = 5$ 2. $\frac{dy}{dx} + y^{1/2} = 2$ 3. $\frac{ydy}{dx} = 5$

3.2.3.4 Leibnitze linear equation

The standard form of a linear equation of the first order, commonly known as **Leibnitz's linear equation,** is

$$\frac{dy}{dx} + Py = Q \text{ where P, Q, are arbitrary functions of x.} \quad \text{... (i)}$$

To solve the equation, multiply both sides by $e^{\int Pdx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int Pdx} + y(e^{\int Pdx} P) = Qe^{\int Pdx} \text{ i.e. } \frac{d}{dx}(ye^{\int Pdx}) = Qe^{\int Pdx}$$

Integrating both sides, we et $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$ as the required solution.

Note. The factor $e^{\int Pdx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor** (I.F.) of the linear equation (i).
So remember the following:

$$\text{I.F.} = e^{\int Pdx}$$

and the solution is $\quad \mathbf{y(I.F.)} = \int Q(\text{I.F.})dx + c.$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.27 Transformation to linear form by substituting $v = y^{1-n}$ of the equation

$\frac{dy}{dt} + p(t)y = q(t)y^n$; $n > 0$ will be

(a) $\frac{dv}{dt} + (1-n)pv = (1-n)q$

(b) $\frac{dv}{dt} + (1-n)pv = (1+n)q$

(c) $\frac{dv}{dt} + (1+n)pv = (1-n)q$

(d) $\frac{dv}{dt} + (1+n)pv = (1+n)q$

[CE, GATE-2005, 2 marks]

Solution: (a)

Given, $\quad \frac{dy}{dt} + p(t)\,y = q(t)\,y^n$; $n > 0$

putting $\quad v = y^{1-n}$

$$\frac{dv}{dt} = (1-n)\,y^{-n}\,\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{(1-n)y^{-n}}\frac{dv}{dt}$$

Substituting in the given differential equation, we get,

$$\frac{1}{(1-n)y^{-n}}\frac{dv}{dt} + p(t)y = q(t)y^n$$

Multiplying by $(1-n)\, y^{-n}$, we get

$$\frac{dv}{dt} + p(t)\,(1-n)y^{1-n} = q(t)\,(1-n)$$

now since $y^{1-n} = v$, we get

$$\frac{dv}{dt} + (1-n)\,pv = (1-n)q$$

(which is linear with v as dependent variable and t as independent variable)

Q.28 The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x$, with the condition that $y = 1$ at $x = 1$, is

(a) $y = \frac{2}{3x^2} + \frac{x}{3}$

(b) $y = \frac{x}{2} + \frac{1}{2x}$

(c) $y = \frac{2}{3} + \frac{x}{3}$

(d) $y = \frac{2}{3x} + \frac{x^2}{3}$

[CE, GATE-2011, 2 mark]

Solution: (d)

$$\frac{dy}{dx} + \frac{y}{x} = x, \quad y(1) = 1$$

This is a linear differential equation $\frac{dy}{dx} + Py = Q$ with $P = \frac{1}{x}$ and $Q = x$

IF = Integrations factor

$$= e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Solution is

$$y.(IF) = \int Q(IF)dx + C$$

$$\Rightarrow \quad y.x = \int (x.x)dx + C$$

$$\Rightarrow \quad yx = \int x^2 dx + C$$

$$\Rightarrow \quad yx = \frac{x^3}{3} + C$$

$$\Rightarrow \quad y = \frac{x^2}{3} + \frac{C}{x}$$

Now y(1) = 1

$$\Rightarrow \frac{1^2}{3} + \frac{C}{1} = 1 \Rightarrow C = \frac{2}{3}$$

So the solution is $y = \frac{x^2}{3} + \frac{2}{3x}$

Q.29 The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1+x)e^{+x^2}$

(b) $(1+x)e^{-x^2}$

(c) $(1-x)e^{+x^2}$

(d) $(1-x)e^{-x^2}$

[ME, GATE-2006, 1 mark]

Solution: (b)

Given equation

$$\frac{dy}{dx} + 2xy = e^{-x2}$$

This is a leibnitz + z linear equation (i.e. a first order linear differential equation)

Integrating factor I.F. $= e^{\int 2x dx} = e^{x2}$

Solution is

$$y(IF) = \int Q(IF)dx + c$$

$$ye^{x^2} = \int e^{-x^2} e^{x^2} dx + c$$

$$ye^{x^2} = x + c$$

at $x = 0,\ y = 1$ (given)

$\therefore$ $1e^0 = 0 + c$

$\Rightarrow$ $c = 1$

So, the solution is $ye^{-x2} = x + 1$

$\Rightarrow$ $y = e^{-x2}(x + 1)$

Q.30 The solution of $x\frac{dy}{dx} + y = x^4$ with the condition $y(1) = \frac{6}{5}$ is

(a) $y = \frac{x^4}{5} + \frac{1}{x}$

(b) $y = \frac{4x^4}{5} + \frac{4}{5x}$

(c) $y = \frac{x^4}{5} + 1$

(d) $y = \frac{x^5}{5} + 1$

[ME, GATE-2009, 2 marks]

Solution: (a)

Given differential equation is

$$x\frac{dy}{dx} + y = x^4$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{y}{x}\right) = x^3 \quad \text{... (i)}$$

Standard form of leibnitz linear equation is

$$\frac{dy}{dx} + Py = Q \quad \text{... (ii)}$$

where P and Q function of x only and solution is given by

$$y(I.F.) = \int Q.(I.F.)\, dx + C$$

where, integrating factor (I.F.) $= e^{\int P dx}$

Here in equation (i), $P = \frac{1}{x}$ and $Q = x^3$

$$I.F. = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Solution $y(x) = \int x^3 . x\, dx + c$

$$yx = \frac{x^5}{5} + c$$

given condition $y(1) = \frac{6}{5}$

means at $x = 1;\ y = \frac{6}{5}$

$$\Rightarrow \quad \frac{6}{5} \times 1 = \frac{1}{5} + c$$

$$\Rightarrow \quad c = \frac{6}{5} - \frac{1}{5} = 1$$

Therefore $yx = \frac{x^5}{5} + 1$

$$\Rightarrow \quad y = \frac{x^4}{5} + \frac{1}{x}$$

Q.31 With initial condition $x(1) = 0.5$, the solution of the differential equation, $t\frac{dx}{dt} + x = t$ is

(a) $x = t - \frac{1}{2}$

(b) $x = t^2 - \frac{1}{2}$

(c) $x = \frac{t^2}{2}$

(d) $x = \frac{t}{2}$

[EC, EE, IN, GATE-2012, 1 mark]

Solution: (d)

The given differential equation is

$t\frac{dx}{dt} + x = t$ with initial condition $x(1) = \frac{1}{2}$ which is same as

$$\frac{dx}{dt} + \frac{x}{t} = 1$$

Which is a linear differential equation

$$\frac{dx}{dt} + Px = Q$$

Where $P = \frac{1}{t}$ and $Q = 1$

$$\text{Integrating factor} = e^{\int P dt} = e^{\int \frac{1}{t} dt}$$

$$= e^{\log_e t} = t$$

Solution is

$$x \cdot (IF) = \int Q \cdot (IF) dt + C$$

$$x \cdot t = \int 1 \cdot t \cdot dt + C$$

$$xt = \frac{t^2}{2} + C$$

$$x = \frac{t}{2} + \frac{C}{t}$$

Put $x(1) = \frac{1}{2}$

$$\Rightarrow \quad \frac{1}{2}+\frac{C}{1} = \frac{1}{2}$$

$$\Rightarrow \quad C = 0$$

So $\boxed{x = \frac{t}{2}}$ is the solution.

Q.32 The matrix form of the linear system $\frac{dx}{dt} = 3x - 5y$ and $\frac{dy}{dt} = 4x + 8y$ is

(a) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(b) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(c) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(d) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (a)

$$\frac{dx}{dt} = 3x - 5y$$

$$\frac{dy}{dt} = 4x + 8y$$

$$\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3x & -5y \\ 4x & +8y \end{bmatrix}$$

Q.33 The general solution of the differential equation $\frac{dy}{dx} = \cos(x + y)$, with c as a constant, is

(a) $y + \sin(x + y) = x + c$

(b) $\tan\left(\frac{x+y}{2}\right) = y + c$

(c) $\cos\left(\frac{x+y}{2}\right) = x + c$

(d) $\tan\left(\frac{x+y}{2}\right) = x + c$

[ME, GATE-2014 : 2 Marks, Set-2]

Solution : (d)

Let $\quad z = x + y$

$$\frac{dz}{dx} = 1+\frac{dy}{dx}$$

$$\therefore \quad \frac{dz}{dx}-1 = \cos z$$

$$\text{or} \quad \int\frac{dz}{1+\cos z} = \int dx$$

$$\text{or} \quad \frac{1}{2}\int \sec^2\left(\frac{z}{2}\right)dz = x + c$$

$$\text{or} \quad \tan\left(\frac{z}{2}\right) = x + c$$

$$\text{or} \quad \tan\left(\frac{x+y}{2}\right) = x + c$$

Q.34 The solution of the initial value problem $\frac{dy}{dx} = -2xy;\ y(0) = 2$ is

(a) $1+e^{-x^2}$
(b) $2e^{-x^2}$
(c) $1+e^{x^2}$
(d) $2e^{x^2}$

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (b)

$$\frac{dy}{dx} = 2xy = 0 \quad \text{...(1)}$$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying I.F. to both side of equation (1)

$$e^{x^2}\left[\frac{dy}{dx} + 2xy\right] = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(e^{x^2}y\right) = 0$$

$$e^{x^2}y = C$$

from the given boundary condition, C = 2

$$\therefore \quad e^{x^2}y = 2$$

$$y = 2e^{-x^2}$$

Q.35 Which ONE of the following is a linear non-homogeneous differential equation, where x and y are the independent and dependent variables respectively?

(a) $\frac{dy}{dx} + xy = e^{-x}$
(b) $\frac{dy}{dx} + xy = 0$
(c) $\frac{dy}{dx} + xy = e^{-y}$
(d) $\frac{dy}{dx} + e^{-y} = 0$

[EC, GATE-2014 : 1 Mark, Set-3]

Solution : (a)

General form of linear differential equation

$\frac{dy}{dx} + py = \theta$ when P and θ can be function of x.

Only option (a) is in this form.

Q.36 The solution for the differential equation $\frac{d^2x}{dt^2} = -9x$ with initial conditions x(0) = 1 and $\left.\frac{dx}{dt}\right|_{t=0} = 1$, is

(a) $t^2 + t + 1$
(b) $\sin 3t + \frac{1}{3}\cos 3t + \frac{2}{3}$
(c) $\frac{1}{3}\sin 3t + \cos 3t$
(d) $\cos 3t + t$

[EE, GATE-2014 : 1 Mark, Set-1]

Solution : (c)

$$\frac{d^2x}{dt^2} = -9x \qquad \frac{d}{dt} = D$$

$$\frac{d^2x}{dt^2} + 9x = 0 \qquad (D^2 + 9)x = 0$$

Auxiliary equation is $m^2 + 9 = 0$

$$m = \pm 3i$$

$$x = C_1 \cos 3t + C_2 \sin 3t \quad \text{...(i)}$$

$$x(0) = 1 \quad \text{i.e. } x \to 1 \text{ when } t \to 0$$

$$\boxed{1 = C_1}$$

$$\frac{dx}{dt} = -3C_1 \sin 3t + 3C_2 \cos 3t \quad \text{...(ii)}$$

$$x'(0) = 1 \quad \text{i.e. } x' \to 1 \text{ when } t \to 0$$

$$1 = 3C_2 \quad \boxed{C_2 = \frac{1}{3}}$$

$\therefore$
$$x = \cos 3t + \frac{1}{3}\sin 3t$$

3.2.3.5 Bernoulli's Equation

The equation $\frac{dy}{dx} + Py = Qy^n$ ... (i)

where P, Q are functions of x, is reducible to the Leibnitz's linear and is usually called the Bernoulli's equation.

To solve (i), divide both sides by y^n, so that $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ... (ii)

Put $y^{1-n} = z$ so that $(1-n)\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (ii) becomes $\quad \frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

or $\quad \frac{dz}{dx} + P(1-n)z = Q(1-n)$,

which is Leibnitz's linear in z and can be solved easily.

ILLUSTRATIVE EXAMPLES

Example:

Solve $\frac{dy}{dx} + y = 4y^3$

Solution:

Dividing throughout by y^3,

$$y^{-3}\frac{dy}{dx} + y^{-2} = 4 \quad \text{... (i)}$$

Put $y^{-2} = z$, so that $-2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (i) becomes $-\frac{1}{2}\frac{dz}{dx}+z=4$

or $\frac{dz}{dx}-2z = -8$ ... (ii)

which is Leibnitz's linear in z.

$$\text{I.F.} = e^{\int -2dx} = e^{-2x}$$

$\therefore$ The solution of (ii) is $z(\text{I.F.}) = \int(-8)(\text{I.F.})dx + c$

$$ze^{-2x} = \int(-8)e^{-2x}dx + c$$

$\Rightarrow$ $y^{-2}e^{-2x} = 4e^{-2x} + c$ $(\because z = y^{-2})$

$\Rightarrow$ $y^{-2} = 4 + ce^{2x}$

$\Rightarrow$ $y = (4 + ce^{2x})^{-1/2}$

3.2.3.6 Exact Differential Equations

1. **Deff.** A differential equation of the form **M(x, y) dx + N(x, y) dy = 0** is said to be **exact** if its left hand member is the exact differential of some function u(x, y) i.e. du = Mdx + Ndy = 0. Its solution, therefore, is u(x, y) = c.
2. **Theorem.** The necessary and sufficient condition for the differential equations Mdx + Ndy = 0 to be exact is
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
3. **Method of solution.** It can be shown that, the equation Mdx + Ndy = 0 becomes
$$d\left[u+\int f(y)dy\right] = 0$$
Integrating $u+\int f(y)dy = 0.$
But $u = \int Mdx$ and f(y) = terms of N not containing x.
$\therefore$ The solution of Mdx + Ndy = 0 is
$\int Mdx + \int$(terms of N not containing x) **dy = c**
(Provides of course that the equation is exact. i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.)
Note: While finding $\int Mdx$, y is treated as constant since we are integrating with respect to x.

ILLUSTRATIVE EXAMPLES

Example:
Solve $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)dy = 0$.

Solution:

Step 1: Test for exactness

Here $M = x^3 + 3xy^2$ and $N = 3x^2y + y^3$

$\therefore$ $\frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x}$

Thus the equation is exact and its solution is

$\int Mdx + \int$(terms of N not containing x)dy $= c$

which is $\int(x^3+3xy^2)dx+\int y^3dy = c$

$$\Rightarrow \quad \frac{x^4}{4}+\frac{3x^2y^2}{2}+\frac{y^4}{4} = c$$

$$\Rightarrow \quad \frac{1}{4}(x^4+6x^2y^2+y^4) = c$$

3.2.3.7 Equations Reducible To Exact Equations

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an integrating factor. The rules for finding integrating factors of the equation Mdx + Ndy = 0 are as given in theorem 1 and 2 below:

In the equation Mdx + Ndy = 0

Theorem 1: if $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ be a function of x only = f(x) say, then $e^{\int f(x)dx}$ is an integrating factor.

Theorem 2: if $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ be a function of y only = f(y) say, then $e^{\int f(y)\,dy}$ is an integrating factor.

ILLUSTRATIVE EXAMPLES

Example: 1

Solve $\quad 2\sin(y^2)\,dx + xy\cos y^2\,dy = 0, \quad y(2) = \sqrt{\frac{\pi}{2}}$

Solution:

Step 1: Here, $\quad M = 2\sin(y^2)$ and $N = xy\cos(y^2)$

Step 2: Test for exactness $\dfrac{\partial M}{\partial y} = 4y\cos(y^2)$ and $\dfrac{\partial N}{\partial x} = y\cos(y^2)$

So $$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

and hence, equation is not exact. So we have to find integrating factor by using either theorem 1 or theorem 2.

Step 3: Find an integrating factor: try theorem 1

Here, $$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{4y\cos y^2 - y\cos y^2}{xy\cos y^2} = \frac{3}{x}$$

Which is function of x only. So theorem 1 can be used.

$$\therefore \quad \text{I.F.} = e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Multiplying throughout by I.F., we get

$2x^3\sin(y^2)\,dx + x^4 y\cos y^2\,dy = 0$

This equation will surely be an exact equation. No need to check that.

Step 4: General solution:

$$\int M dx + \int (\text{terms of N containing x}) dy = c$$

Which is $\int 2x^3 \sin(y^2) dx + \int 0 dy = c$

$$\frac{1}{2}x^4 \sin(y^2) = c$$

Step 5: Now to find the particular solution of the initial value problem:

Since $y(2) = \sqrt{\frac{\pi}{2}}$

$$\Rightarrow \quad \frac{1}{2} \cdot 2^4 \sin\frac{\pi}{2} = c$$

$$\Rightarrow \quad c = 8$$

So particular solution is $\frac{1}{2}x^4 \sin(y^2) = 8$

or $x^4 \sin(y^2) = 16$

Example: 2

Solve $(xy^3 + y)\, dx + 2(x^2y^2 + x + y^4)\, dy = 0$.

Solution:

Here $M = xy^3 + y,\ N = 2(x^2y^2 + x + y^4)$

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{dx} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)}(4xy^2 + 2 - 3xy^2 - 1)$$

$$= \frac{1}{y}, \text{ which is a function of y alone.}$$

$$\therefore \quad \text{I.F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0$, which is an exact equation.

$\therefore$ The solution is $\frac{1}{2}x^2y^4 + xy^2 + \frac{1}{3}y^6 = c$.

3.2.4 Orthogonal Trajectories

3.2.4.1 Definitions

Two families of curves such that every member of either family cuts each member of the other family at right angles are called orthogonal trajectories of each other.

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems.

For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of equipotential curves and vice versa.

In fluid flow, the stream lines and the equipotential lines are orthogonal trajectories.

ILLUSTRATIVE EXAMPLES

Example: 1

Find the orthogonal trajectory of family of curves xy = constant.

Solution:

Given family of curves $xy = c$ ...(i)

Differentiate w.r.t. 'x'

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

Now replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\Rightarrow \quad x\frac{dx}{dy} = y$$

By variable separable,

$$\int x dx = \int y dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + k$$

$\Rightarrow$ $x^2 - y^2 = k_1$ is the orthogonal tragectory of given family of curves

3.2.4.2 Orthogonal trajectory of polar curves

Example: 2

Find the orthogonal trajectory of family of curves $r^n = a^n \sin n\theta$

Solution:

Given family of curves $r^n = a^n \sin\theta$ ..(i)

Differentiate w.r.t. 'θ' and eliminate 'a'

$$nr^{n-1}\frac{dr}{d\theta} = a^n \cos nq \times n \quad ...(ii)$$

Divided equation (ii) by equation (i)

$$\frac{nr^{n-1}\frac{dr}{d\theta}}{r^n} = \frac{a^n \cos n\theta \times n}{a^n \sin n\theta}$$

$$\frac{dr}{d\theta}\frac{1}{r} = \cot n\theta \quad ...(iii)$$

Differential equation represents given family of curves.

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot n\theta$$

$$-r\frac{d\theta}{dr} = \cot n\theta$$

$$\int \frac{1}{r} dr = -\int \tan n\theta \, d\theta$$

$$\log r = -\frac{\log \sec n\theta}{n} + \log c$$

$$\log r^n = \log[c^n \cos n\theta]$$

$$\Rightarrow \quad r^n = an \cos n\theta$$

is the required orthogonal trajectory.

3.2.4.3 Newton's Law of Cooling

Definitions

The temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

The differential equation is $\frac{d\theta}{dt} = -k(\theta - \theta_s)$

by variable separable $\int \frac{d\theta}{\theta - \theta_s} = \int -kdt$

$\Rightarrow \quad \log(\theta - \theta_s) = -kt + \log c$

$\Rightarrow \quad \theta - \theta_s = ce^{-kt}$

is the solution of Newton's law of cooling.

Example: 3

A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of air being 40°C. What will be the temperature of body after 40 minutes from the original?

Solution:

According to Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - 40)$$

$$\int \frac{d\theta}{\theta - 40} = -\int kdt$$

$\Rightarrow \quad \log(\theta - 40) = -kt + \log c$

$\Rightarrow \quad \theta - 40 = ce^{-kt} \quad$ ...(i)

Put $t = 0,\ \theta = 80°$ in equation (i)

We get, $c = 40$

Put, $t = 20$ min, $\theta = 60°$

Then, $k = \frac{1}{20}\log 2$

By equation (i), $\theta = 40 + 40e^{\left(-\frac{1}{2}\log 2\right)}$

Put, $t = 40$ min, then $\theta = 50°C$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.37 A body originally at 60°C cools down to 40°C in 15 minutes when kept in air at a temperature of 25°C. What will be the temperature of the body at the end of 30 minutes?

(a) 35.2°C
(b) 31.5°C
(c) 28.7°C
(d) 15°C

[CE, GATE-2007, 2 marks]

Solution: (b)

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \quad \text{(Newton's law of cooling)}$$

This is in variable seperable form separating the variables, we get,

$$\frac{d\theta}{\theta - \theta_0} = -kdt$$

$$\int \frac{d\theta}{\theta - \theta_0} = \int -k\,dt$$

$\Rightarrow$ $\ln(\theta - \theta_0) = -kt + C_1$

$\Rightarrow$ $\theta - \theta_0 = C.e^{-kt}$ (where $C = e^{c_1}$)

$\theta = \theta_0 + C.e^{-kt}$

given, $\theta_0 = 25°C$

Now at $t = 0, \theta = 60°$

$60 = 25 + C.e^0$

$\Rightarrow$ $C = 35$

$\therefore$ $\theta = 25 + 35\,e^{-kt}$

at t = 15 minutes $\theta = 40°C$

$\therefore$ $40 = 25 + 35e^{(-k \times 15)}$

$\Rightarrow$ $e^{-15k} = \frac{3}{7}$ ... (i)

Now at t = 30 minutes

$$\theta = 25 + 35\,e^{-30k} = 25 + 35\,(e^{-15k})^2$$

Now substituting $e^{-15k} = \frac{3}{7}$ from (i), we get,

$$\theta = 25 + 35 \times \left(\frac{3}{7}\right)^2 = 31.428°C \approx 31.5°C$$

3.2.4.4 Law of Growth

The rate of change amount of a substance with respect to time is directly proportional to the amount of substance present.

i.e. $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = kx \quad (k > 0)$

$\int \frac{dx}{x} = \int k\,dt$

$\Rightarrow$ $\log x = kt + \log c$

$\Rightarrow$ $x = ce^{kt}$ is solution of law of growth

Example: 4

The number N of a bacteria in a culture of grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours?

Solution:

According to law of growth, $\frac{dN}{dt} \propto N$

Solution is $N = ce^{kt}$ ...(i)

Put N = 100 and t = 0 in equation (i)

We get, $c = 100$

Then, $N = 100\,e^{kt}$ ...(ii)

Put N = 332, t = 1 in equation (ii)

$$332 = 100e^k$$
$$e^k = 3.32$$

Put $t = \frac{3}{2}$ in equation (ii)

Then, $$N = 100e^{\frac{3}{2}k} = 100(3.32)^{3/2} \simeq 605$$

3.2.4.5 Law of Decay

Definitions

The rate of change of amount of substance is directly proportional to the amount of substance present.

i.e. $$\frac{dx}{dt} \propto x$$

The differential equation is

$$\frac{dx}{dt} = -kx \quad (k > 0)$$
$$\int \frac{dx}{x} = -\int k\,dt$$
$$\Rightarrow \quad \log x = -kt + \log c$$
$$\Rightarrow \quad x = ce^{-kt}$$ is solution of law of decay.

Example: 5

If 30% of radio active substance dissappeared in 10 days. How long will take for 90% of it to disappear?

Solution:

According to law of decay

$$x = ce{-kt} \quad \text{...(i)}$$

Put x = 100, t = 0

$$100 = ce^{-k(0)}$$

We get, $$c = 100$$

Then, $$x = 100e^{-kt} \quad \text{...(ii)}$$

Put x = 70, t = 10 in equation (ii)

Then, $$70 = 100\,e^{10k}$$
$$k = \frac{1}{10} ln\left[\frac{7}{10}\right]$$

∴ Equation (ii) becomes $$x = 100\,e^{\frac{1}{10} ln\left(\frac{7}{10}\right)} \quad \text{...(iii)}$$

Put, x = 10 in equation (iii)

$$10 = 100\,e^{\frac{1}{10} ln\left(\frac{7}{10}\right)}$$
$$\frac{t}{10} ln(0.7) = ln\left(\frac{1}{10}\right)$$
$$t = \frac{-10 ln 10}{ln(0.7)} = 64.5 \text{ days}$$

3.3 LINEAR DIFFERENTIAL EQUATIONS (OF n^{TH} ORDER)

3.3.1 Definitions

Linear differential equations are those in which the dependent variable its derivatives occur only in the first degree and are not multiplied together. The general linear differential equation of the n^{th} order is of the form

$$\frac{d^n y}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+p_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+p_n y = X$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear Differential Equations with Constant Coefficients are of the form

$$\frac{d^n y}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+k_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+k_n y = X$$

where $k_1, k_2 \ldots, k_n$ are constants and X is a function of x only. Such equations are most important in the study of electromechanical vibrations and other engineering problems.

1. **Theorem:** If y_1, y_2 are only two solutions of the equations

$$\frac{d^n y}{dx^n}+\frac{d^{n-1}y}{dx^{n-1}}+k_2\frac{d^{n-2}}{dx^{n-2}}+\ldots+k_n y = 0 \quad \ldots (i)$$

Then $c_1y_1 + c_{21}y_2$ (= u) is also its solution,

since it can be easily shown by differentiating is that $\frac{d^n u}{dx^n}+k_1\frac{d^{n-1}u}{dx^{n-1}}+\ldots\ldots+k_n u = 0$ ... (ii)

2. Since the general solution of a differential equation of the nth order contains n arbitrary constants, if follows, from above, that if $y_1, y_2, y_3, \ldots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \ldots + c_ny_n$(= u) is its complete solution.

3. If y = v be any particular solution of

$$\frac{d^n y}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+k_n y = X \quad \ldots (iii)$$

then
$$\frac{d^n v}{dx^n}+k_1\frac{d^{n-1}v}{dx^{n-1}}+\ldots+k_n v = X \quad \ldots (iv)$$

Adding (ii) and (iv), we have $\frac{d^n(u+v)}{dx^n}+k_1\frac{d^{n-1}(u+v)}{dx^{n-1}}+\ldots+k_n(u+v) = X$

This shows that y = u + v is the complete solution of (iii).

The part u is called the **complementary function (C.F.)** and the part v is called the **particular integral (P.I.)** of (iii).

∴ The complete solution (C.S.) of (iii) is **y = C.F. + P.I.**

Thus in order to solve the equation (iii), we have to first find the C.F. i.e., the complementary function of (i), and then the P.I., i.e. a particular solution of (iii).

Operator D Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc., so that

$\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y, \frac{d^3x}{dx^3} = D^3y$ etc., the equation (iii) above can be written in the symbolic form

$$(D^n + k_1D^{n-1} + \ldots + k_n)y = X,$$

i.e. $f(D)y = X,$

where $f(D) = D^n + k_1 D^{n-1} + ... + k_n$, i.e. a polynomial in D.

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e. f(D) can be factorised by ordinary rules of algebra and the factors may be taken in any order. For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)y$$

$$= (D + 3)(D - 1)y \text{ or } (D - 1)(D + 3)y.$$

3.3.2 Rules for Finding The Complementary Function

To solve the equation $\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}}{dx^{n-2}} + ... + k_n y = 0$ ... (i)

where k's are constants.

The equation (i) in symbolic form is

$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + ... + k_n)y = 0$ ... (ii)

Its symbolic co-efficient equated to zero i.e.

$D^n + k_1 D^{n-1} + k_2 D^{n-2} + ... + k_n = 0$

is called the auxiliary equation (A.E.). Let $m_1, m_2, ..., m_n$ be its roots. Now 4 cases arise.

Case I. If **all the roots be real and different**, then (ii) is equivalent to

$(D - m_1)(D - m_2) ... (D - m_n)y = 0$ ... (iii)

Now (iii) will be satisfied by the solution of $(D - m_n)y = 0$, i.e. by $\frac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ Its solution is $ye^{-m_n x} = c_n$, i.e. $y = c_n e^{m_n x}$

Similarly, since the factors in (iii) can be taken in any order, it will be satisfied by the solutions of $(D - m_1)y = 0$, $(D - m_2) = 0$ etc., i.e. by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (i) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$ ... (iv)

Case II. If **two roots are equal** (i.e. $m_1 = m_2$), then (iv) becomes

$$y = (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

$$y = Ce^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

[$\because c_1 + c_2$ = one arbitrary constant C]

It has only n – 1 arbitrary constants and is, therefore, not the complete solution of (i). In this case, we proceed as follows:

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1)z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is Leibnitz's linear in z and I.F. $= e^{-m_1 x}$

$\therefore$ Its solution is $ze^{-m_1x} = c_1$ or $z = c_1e^{m_1x}$

Thus
$$(D - m_1)\,y = z = c_1e^{m_1x} \text{ or } \frac{dy}{dx} - m_1y = c_1e^{m_1x} \quad \text{... (v)}$$

Its I.F. being e^{-m_1x}, the solution of (v) is

$$ye^{-m_1x} = \int c_1e^{m_1x}e^{-m_1x}dx + c_2$$

$$\Rightarrow \quad y = (c_1x + c_2)e^{m_1x}$$

Thus the complete solution of (i) is $y = (c_1x + c_2)\,e^{m_1x} + c_3e^{m_3x} + ... + c_ne^{m_nx}$

If, however, the A.E. has three equal roots (i.e. $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1x^2 + c_2x + c_3)e^{m_1x} + c_4e^{m_4x} + ... + c_ne^{m_nx}$$

Case III. If **one pair of roots be imaginary,** i.e.

$$m_1 = \alpha + i\beta,$$
$$m_2 = \alpha - i\beta,$$

then the complete solution is

$$y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} + c_3e^{m_3x} + ... + c_ne^{m_nx}$$

$$= e^{\alpha x}(c_1e^{i\beta x} + c_2e^{-i\beta x}) + c_3e^{m_3x} + ... + c_ne^{m_nx}$$

$$= e^{\alpha x}[c_1\,(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] + c_3e^{m_3x} + ... + c_ne^{m_nx}$$

[$\because$ by Euler's Theorem, $e^{i\theta} = \cos\theta + i\sin\theta$]

$$= e^{\alpha x}\,(C_1\cos\beta x + C_2\sin\beta x) + c_3e^{m_3x} + ... + c_ne^{m_nx}$$

where $C_1 = c_1 + c_2$

and $C_2 = i(c_1 - c_2)$.

Case IV. If **two pair of imaginary roots be equal** i.e.

$$m_1 = m_2 = \alpha + i\beta,$$
$$m_3 = m_4 = \alpha - i\beta,$$

then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1x + c_2)\cos\beta x + (c_3x + c_4)\sin\beta x] + ... + c_ne^{m_nx}.$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.38 The solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 0$; $y(0) = 1$, $\frac{dy}{dx}\left(\frac{\pi}{4}\right) = 0$ in the range $0 < x < \frac{\pi}{4}$ is given by

(a) $e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)$

(b) $e^{x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

(c) $e^{-4x}\left(\cos x - \frac{1}{4}\sin x\right)$

(d) $e^{-4x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

[CE, GATE-2005, 2 marks]

Solution: (a)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 0$$

$$y(0) = 1$$

$$\frac{dy}{dx}\left(\frac{\pi}{4}\right) = 0$$

This is a linear differential equation

$$D^2 + 2D + 17 = 0$$

$$D = -1 \pm 4i$$

$$\therefore \quad y = C_1 e^{(-1+4i)x} + C_2 e^{(-1-4i)x} = e^{-x} C_1 e^{4xi} + C_2 e^{-4x}$$

$$= e^{-x}[C_1(\cos 4x + i\sin 4x)] + C_2[\cos(-4x) + i\sin(-4x)]$$

$$= e^{-x}[(C_1 + C_2)\cos 4x + (C_1 - C_2)\, i \sin 4x)]$$

Let $C_1 + C_2 = C_3$ and $(C_1 - C_2)\, i = C_4$

$$y = e^{-x}(C_3 \cos 4x + C_4 \sin 4x)$$

since $y(0) = 1$

$$1 = e^{-0}(C_3 \cos 0 + C_4 \sin 0)$$

$\Rightarrow$ $C_3 = 1$

$\Rightarrow$

$$\frac{dy}{dx} = e^{-x}(-4C_3 \sin 4x + 4C_4 \cos 4x) - e^{-x}[C_3 \cos 4x + C_4 \sin 4x]$$

$$= e^{-x}[(-4C_3 - C_4)\sin 4x + (4C_4 - C_3)\cos 4x]$$

$$\frac{dy}{dx} \text{ at } x = \frac{\pi}{4} \text{ is } 0$$

$$\therefore \quad -(4C_4 - C_3)\, e^{-\pi/4} = 0$$

$$4C_4 = C_3$$

$$C_4 = \frac{C_3}{4} = \frac{1}{4}$$

$$\therefore \quad C_3 = 1 \text{ and } C_4 = \frac{1}{4}$$

$$y = e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)$$

Q.39 The general solution of $\frac{d^2y}{dx^2} + y = 0$ is

(a) $y = P\cos x + Q\sin x$
(b) $y = P\cos x$
(c) $y = P\sin x$
(d) $y = P\sin^2 x$

[CE, GATE-2008, 1 mark]

Solution: (a)

$$\frac{d^2y}{dx^2} + y = 0$$

$$D^2 + 1 = 0$$

$$D = \pm i = 0 \pm 1i$$

$\therefore$ General solution is

$$y = e^{0x}[C_1 \cos(1 \times x) + C_2 \sin(1 \times x)]$$

$$= C_1 \cos x + C_2 \sin x = P\cos x + Q\sin x$$

where P and Q are some constants.

Q.40 The solution to the ordinary differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ is

(a) $y=c_1e^{3x}+c_2e^{-2x}$
(b) $y=c_1e^{3x}+c_2e^{2x}$
(c) $y=c_1e^{-3x}+c_2e^{2x}$
(d) $y=c_1e^{-3x}+c_2e^{-2x}$

[CE, GATE-2010, 2 marks]

Solution: (c)

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y = 0$$
$$D^2+D-6 = 0$$
$$(D+3)(D-2) = 0$$
$$D = -3 \text{ or } D=2$$

∴ Solution is $y = c_1e^{-3x}+c_2e^{2x}$

Statement for Linked Answer Questions 41 and 42.

The complete solution of the ordinary differential quation $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$ is $y=c_1e^{-x}+c_2e^{-3x}$.

Q.41 Then, p and q are

(a) $p=3, q=3$
(b) $p=3, q=4$
(c) $p=4, q=3$
(d) $p=4, q=4$

[ME, GATE-2005, 2 marks]

Solution: (c)

Given equation is

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy = 0$$
$$(D^2+pD+q)y = 0$$

∴ $D^2+pD+q = 0$

Its solution is $y = C_1e^{-x}+C_2e^{-3x}$

So the roots of $D^2+pD+q=0$ are $\alpha=-1$ and $\beta=-3$

Sum of roots $= -p = -1-3 \Rightarrow p=4$

Product of roots $= q = (-1)(-3) \Rightarrow q=3$

Q.42 Which of the following is a solution of the differential equation $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+(q+1)=0$?

(a) e^{-3x}
(b) xe^{-x}
(c) xe^{-2x}
(d) x^2e^{-2x}

[ME, GATE-2005, 2 marks]

Solution: (c)

Given equation is

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+(q+1) = 0$$

$\Rightarrow$ $[D^2+PD+(q+1)]y = 0$

Put $p = 4$

and $q = 3$

∴ $(D^2+4D+4)y = 0$

$$D^2+4D+4 = 0$$
$$(D+2)^2 = 0$$

$\therefore$ $D = -2, -2 \Rightarrow y = (c_1x + c_2)\,e^{-2x}$

out of choices given, $y = x\,e^{-2x}$

is the only answer in the required form (i.e. $(c_1x + c_2)e^{-2x}$ putting $c_1 = 1$ and $c_2 = 0$)

Q.43 Given that $\ddot{x} + 3x = 0$, and $x(0) = 1$, $\dot{x}(0) = 0$, what is $x(1)$?

(a) –0.99
(b) –0.16
(c) 0.16
(d) 0.99

[ME, GATE-2008, 1 mark]

Solution: (b)

$$\ddot{x} + 3x = 0$$

Auxiliary equation is $D^2 + 3 = 0$

i.e. $D = \pm\sqrt{3}\,i$

$\therefore$ $x = A\cos\sqrt{3}t + B\sin\sqrt{3}t$

at $t = 0, x = 1$

$\Rightarrow$ $A = 1$

Now, $\dot{x} = \sqrt{3}(B\cos\sqrt{3}t - A\sin\sqrt{3}t)$

At $t = 0, \dot{x} = 0$

$\Rightarrow$ $B = 0$

So, $x = \cos\sqrt{3}t$

$$x(1) = \cos\sqrt{3} = 0.99$$

Q.44 It is given that $y'' + 2y' + y = 0$, $y(0) = 0$, $y(1) = 0$. What is $y(0.5)$?

(a) 0
(b) 0.37
(c) 0.62
(d) 1.13

[ME, GATE-2008, 2 marks]

Solution: (a)

$$y'' + 2y' + y = 0$$

$$(D^2 + 2D + 1)y = 0$$

$\Rightarrow$ $D^2 + 2D + 1 = 0$

$\Rightarrow$ $(D + 1)^2 = 0$

$\Rightarrow$ $D = -1, -1$

$\therefore$ $y = (C_1 + C_2x)e^{-x}$

$y(0) = 0 \Rightarrow$ $0 = (C_1 + C_2(0))e^{-0}$

$\Rightarrow$ $C_1 = 0$

$y(1) = 0 \Rightarrow$ $0 = (C_1 + C_2)\,e^{-1}$

$\Rightarrow$ $C_1 + C_2 = 0$

$\Rightarrow$ $C_2 = 0$

$\therefore$ $y = (0 + 0x)e^{-x} = 0$ is the solution

$\therefore$ $y(0.5) = 0$

Q.45 For the differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$ with initial conditions $x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$, the solution is

(a) $x(t) = 2e^{-6t} - e^{-2t}$
(b) $x(t) = 2e^{-2t} - e^{-4t}$
(c) $x(t) = -e^{-6t} + 2e^{-4t}$
(d) $x(t) = e^{-2t} + 2e^{-4t}$

[EE, GATE-2010, 2 marks]

Solution: (b)

Given, $$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$$

$$x(0) = 1 \text{ and } \left.\frac{dx}{dt}\right|_{t=0} = 0$$

$$D^2 + 6D + 8 = 0$$

$$(D+4)(D+2) = 0$$

$$D = -2 \text{ and } D = -4$$

∴ Solution is $x = C_1 e^{-2t} + C_2 e^{-4t}$

Since, $x(0) = 1$

We have $C_1 + C_2 = 1$ ... (i)

$$\frac{dx}{dt} = -2C_1 e^{-2t} - 4C_2 e^{-4t}$$

Since, $\left[\frac{dx}{dt}\right]_{t=0} = 0$, we have

$$-2C_1 - 4C_2 = 0 \quad \text{... (ii)}$$

Solving (i) and (ii) we have, $C_1 = 2$ and $C_2 = -1$

So the solution is $x(t) = 2e^{-2t} - e^{-4t}$

Q.46 The solution to the differential equation $\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0$ is where k is constant, subjected to the boundary conditions $u(0) = 0$ and $u(L) = U$, is

(a) $u = U\frac{x}{L}$

(b) $u = U\left(\frac{1-e^{kx}}{1-e^{kL}}\right)$

(c) $u = U\left(\frac{1-e^{-kx}}{1-e^{-kL}}\right)$

(d) $u = U\left(\frac{1+e^{kx}}{1+e^{kL}}\right)$

[ME, GATE-2013, 2 Marks]

Solution: (b)

Given differential equation $(D^2 - kD)u = 0$

It is linear differential equation with constant coefficient

∴ General solution is $u = CF + PI$

CF: It is given by $f(m) = m^2 - mk = 0 \Rightarrow m = 0, m = k$

∴ $CF = C_1 e^{0.x} + C_2 e^{kx}$

∴ $u = u_{CF} = C_1 + C2e^{kx}$ ...(i)

Put, $x = 0, u = 0$

We get, $C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$

Put, $x = L$ and $u = U$

$$U = C_1 + C_2 e^{kL}$$

$$U = -C_2 + C_2 e^{kL}$$

$$C_2 = \frac{U}{(e^{kL}-1)} \Rightarrow C_1 = \frac{U}{1-e^{kL}}$$

∴ $$u = \frac{U}{1-e^{kL}} - \frac{U}{1-e^{kL}}e^{kx}$$

$$u = U\left[\frac{1-e^{kx}}{1-e^{kL}}\right]$$

Q.47 The maximum value of the solution y(t) of the differential equation $y(t) + \ddot{y}(t) = 0$ with initial conditions $\dot{y}(0) = 1$ and $y(0) = 1$, for $t \geq 0$ is

(a) 1
(b) 2
(c) π
(d) $\sqrt{2}$

[IN, GATE-2013 : 2 marks]

Solution: (d)

$$y(t) + \ddot{y}(t) = 0$$
$$1 + D^2 = 0$$
$$\therefore \quad D = \pm i$$
$$\therefore \quad y = C_1 e^{ix} + C_2 e^{-ix}$$
$$= A\cos x + B\sin x$$
$$y(0) = 1$$
$$\therefore \quad 1 = A \times 1 + B \times 0$$
$$A = 1$$
$$\dot{y} = -A\sin x + B\cos x$$
$$\dot{y}(0) = 1$$
$$\therefore \quad 1 = -A \times 0 + B \times 1$$
$$\therefore \quad B = 1$$

So, $y = \cos x + \sin x$

for maxima,

$$y' = -\sin x + \cos x = 0$$
$$\therefore \quad \sin x = \cos x$$
$$\therefore \quad x = 45^\circ$$
$$y'' = -\cos x - \sin x$$

$\therefore$ $y'' < 0$ for $x = 45^\circ$ $\therefore$ maxima

$$y(\max) = \cos 45^\circ + \sin 45^\circ = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Q.48 A solution of the ordinary differential equation $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is such that $y(0) = 2$ and $y(1) = -\frac{1-3e}{e^3}$. The value of $\frac{dy}{dt}(0)$ is ______.

[EE, GATE-2015 : 2 Marks, Set-1]

Solution: (–3)

$$D^2 + 5D + 6 = 0$$
$$D = -2, -3$$
$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Given, $y(0) = 2$

$$\Rightarrow \quad c_1 + c_2 = 2 \quad \text{...(i)}$$
$$y(1) = -\left(\frac{1-3e}{e^3}\right)$$
$$\Rightarrow \quad \frac{c_1}{e^2} + \frac{c_2}{e^3} = -\left(\frac{1-3e}{e^3}\right)$$

$\Rightarrow \quad ec_1 + c_2 = 3e - 1$ ...(ii)

Now solving equation (i) and (ii), we get

$$c_1 = 3$$
$$c_2 = -1$$

Substituting in $y(t)$, we get

$$y(t) = 3e^{-2t} - e^{-3t}$$

Now, $$\frac{dy}{dt} = -6e^{-2t} + 3e^{-3t}$$

$$\left(\frac{dy}{dt}\right)_{t=0} = -6 + 3 = -3$$

Q.49 A solution of the following differential equation is given by $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

(a) $y = e^{2x} + e^{-3x}$
(b) $y = e^{2x} + e^{3x}$
(c) $y = e^{-2x} + e^{3x}$
(d) $y = e^{-2x} + e^{-3x}$

[EC, GATE-2005, 1 mark]

Solution: (b)

$$\text{A.E.} \Rightarrow D^2 - 5D + 6 = 0$$
$$(D-2)(D-3) = 0$$
$$D = 2, 3$$
$$\therefore \quad y = e^{2x} + e^{3x}$$

Q.50 A function n(x) satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$ where L is a constant. The boundary conditions are : $n(0) = K$ and $n(\infty) = 0$. The solution to this equation is

(a) $n(x) = K\exp(x/L)$
(b) $n(x) = K\exp(-x/\sqrt{L})$
(c) $n(x) = K^2\exp(-x/L)$
(d) $n(x) = K\exp(-x/L)$

[EC, GATE-2010, 1 mark]

Solution: (d)

$$\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$$

$$\Rightarrow \quad D^2 - \frac{1}{L^2} = 0$$

$$\Rightarrow \quad D^2 = \frac{1}{L^2} \qquad D = \pm\frac{1}{L}$$

$\therefore$ Solution is

$$n(x) = C_1 e^{-1/L\,x} + C_2 e^{1/L\,x}$$
$$n(0) = C_1 + C_2 = K$$
$$n(\infty) = C_1 e^{-\infty} + C_2 e^{\infty} = 0$$
$$\Rightarrow \quad C_2 e^{\infty} = 0$$
$$\Rightarrow \quad C_2 = 0$$
$$\therefore \quad C_1 = K$$

$\therefore$ The solution is $\quad n(x) = K e^{-1/L\,x}$

Q.51 A system described by a linear, constant coefficient, ordinary, first order differential equation has an exact solution given by y(t) for t > 0, when the forcing function is x(t) and the initial condition is y(0). If one wishes to modify the system so that the solution becomes -2y(t) for t > 0, we need to

(a) change the initial condition to –y(0) and the forcing function to 2x(t)
(b) change the initial condition to 2y(0) and the forcing function to –x(t)
(c) change the initial condition to $j\sqrt{2}\,y(0)$ and the forcing function to $j\sqrt{2}\,x(t)$
(d) change the initial condition to –2y(0) and the forcing function to –2x(t)

[EC, GATE-2013, 2 Marks]

Solution: (d)

$$\frac{dy(t)}{dt} + ky(t) = x(t)$$

Taking Laplace transform of both sides, we have sY(s) – y(0) + kY(s) = X(s)

$$Y(s)[s + k] = X(s) + y(0)$$

$$\Rightarrow \quad Y(s) = \frac{X(s)}{s+k} + \frac{y(0)}{s+k}$$

Taking inverse Laplace transform, we have

$$y(t) = e^{-kt}x(t) + y(0)e^{-kt}$$

So if we want –2y(t) as a solution both x(t) and y(0) has to be multiplied by –2; hence change x(t) by –2x(t) and y(0) by –2y(0).

Q.52 The solution of the differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ with $y(0) = y'(0) = 1$ is

(a) $(2 - t)e^t$
(b) $(1 + 2t)e^{-t}$
(c) $(2 + t)e^{-t}$
(d) $(1 - 2t)e^t$

[EC, GATE-2015 : 2 Marks, Set-1]

Solution: (b)

$$(D^2 + 2D + 1) = 0$$
$$D = -1, -1$$
$$y(t) = (C_1 + C_2\, t)\, e^{-t}$$
$$y'(t) = C_2\, e^{-t} + (C_1 + C_2 t)\,(-e^{-t})$$
$$y(0) = y'(0) = 1$$

From here, $C_1 = 1,\ C_2 = 2$

$$\Rightarrow \quad y(t) = (1 + 2t)e^{-t}$$

Q.53 Consider the differential equation $\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 0$. Given $x(0) = 20$ and $x(1) = 10/e$, where e = 2.718, the value of $x(2)$ is____.

[EC, GATE-2015 : 2 Marks, Set-3]

Solution: (0.8553)

$$D^2 + 3D + 2 = 0$$
$$D = -1, -2$$
$$x(t) = C_1\, e^{-t} + C_2\, e^{-2t}$$
$$x(1) = \frac{10}{e} = C_1 e^{-1} + C_2\, e^{-2}$$

$$C_1 + C_2 e^{-1} = 10 \quad \text{...(i)}$$
$$C_1 + C_2 = 20 \quad \text{...(ii)}$$

From here, $C_1 = \dfrac{10e - 20}{e-1}$; $C_2 = \left(\dfrac{10e}{e-1}\right)$

$$x(2) = \left(\frac{10e-20}{e-1}\right)e^{-2} + \left(\frac{we}{e-1}\right)e^{-4} = 0.8566$$

Q.54 A function $y(t)$, such that $y(0) = 1$ and $y(1) = 3e^{-1}$, is a solution of the differential equation $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$. Then $y(2)$ is

(a) $5e^{-1}$
(b) $5e^{-2}$
(c) $7e^{-1}$
(d) $7e^{-2}$

[EE, 2016 : 1 Mark, Set-1]

Solution: (b)

Auxiliary equation,

$$m^2 + 2m + 1 = 0$$
$$m = -1, -1$$
$$y = (c_1 + c_2 t)\, e^{-t}$$
$$y(0) = 1$$
$$\Rightarrow \quad c_1 = 1$$
$$y = (1 + c_2 t)e^{-t}$$
$$y(1) = 3e^{-1}$$
$$\Rightarrow \quad (1 + (c_2)e^{-3} = 3e^{-3}$$
$$c_2 = 2$$
$$y = (1 + 2t)e^{-t}$$
$$y(2) = 5e^{-2}$$

Q.55 The solution of the differential equation, for $t > 0$, $y''(t) + 2y'(t) + y(t) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$, is ($u(t)$ denotes the unit step function),

(a) $te^{-t}u(t)$
(b) $(e^{-t} - te^{-t})u(t)$
(c) $(-e^{-t} + te^{-t})u(t)$
(d) $e^{-t}u(t)$

[EE, 2016 : 1 Mark, Set-2]

Solution: (a)

The differential equation is

$y''(t) + 2y'(t) + y(t) = 0$

So, $(s^2Y(s) - sy(0) - y'(0)) + 2[sY(s) - y(0)] + Y(s)$

So, $$Y(s) = \frac{sy(0) + y'(0) + 2y(0)}{(s^2 + 2s + 1)}$$

Given that, $y'(0) = 1$, $y(0) = 0$

So, $$Y(s) = \frac{1}{(s+1)^2}$$

So, $$y(t) = te^{-t}\, u(t)$$

Q.56 Let $y(x)$ be the solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ with initial conditions $y(0) = 0$ and $\left.\frac{dy}{dx}\right|_{x=0} = 1$. Then the value of $y(1)$ is ____.

[EE, 2016 : 2 Marks, Set-2]

Solution:

A.E. $m^2 - 4m + 4 = 0$

$$m = 2, 2$$

$$y = (C_1 + C_2x)\,e^{2x}$$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y = C_2 x\,e^{2x}$$

$$y' = C_2e^{2x} + 2C_2x\,e^{2x}$$

$$y'(0) = 1$$

$$\Rightarrow \quad C_2 = 1$$

$$y = x\,e^{2x}$$

$$y(1) = e^2 = 7.38$$

3.3.3 Inverse Operator

1. Definition, $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants, which when operated upon by f(D) gives X.

i.e. $$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ satisfies the equation f(D)y = X and is, therefore, its particular integral.

Obviously. f(D) and 1/f(D) are inverse operators.

2. $$\frac{1}{D}X = \int X dx$$

Let $$\frac{1}{D}X = y$$

Operating by D, $$D\frac{1}{D}X = Dy$$

i.e. $$X = \frac{dy}{dx}$$

integrating w.r.t. x, $$y = \int X dx$$

Thus $$\frac{1}{D}X = \int X dx$$

3. $$\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.$$

Let $$\frac{1}{D-a}X = y. \quad \text{...(ii)}$$

Operating by D – a,

$$(D-a)\cdot\frac{1}{D-a}X = (D-a)y.$$

or

$$X = \frac{dy}{dx} - ay, \text{ i.e. } \frac{dy}{dx} - ay = X$$

which is a Leibnitz's linear equations.

∴ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$

no constant being added as (ii) doesn't contain any constant.

Thus,

$$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

3.3.4 Rules For Finding The Particular Integral

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + ... + k_n y = X$

which in symbolic form is $(D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n)y = X$.

∴

$$\text{P.I.} = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n}X.$$

Case I. When $X = e^{ax}$

Since

$$De^{ax} = ae^{ax}$$

$$D^2e^{ax} = a^2e^{ax}$$

....

....

$$D^ne^{ax} = a^ne^{ax}$$

$$(D^n + k_1D^{n-1}...+k_n)e^{ax} = (a^n + k_1a^{n-1}...+k_n)e^{ax}$$

i.e.

$$f(D)\,e^{ax} = f(a)\,e^{ax}$$

Operating on both sides by

$$\frac{1}{f(D)}\cdot f(D)e^{ax} = \frac{1}{f(D)}\cdot f(a)e^{ax}$$

or

$$e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

∴ by ÷ f(a)

∴

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \text{ provided } f(a) \neq 0 \quad ...(i)$$

If f(a) = 0, the above rule fails and we proceed further.

It can be proved that in that case,

$$\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax} \quad ...(ii)$$

If f′(a) = 0, then applying (2) again, we get $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$, provided $f''(a) \neq 0$ ...(iii)

and so on.

ILLUSTRATIVE EXAMPLES

Example 1. Solve

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y = 5e^{3x}$$

Solution:

$$(D^2+6D+9)y = 5e^{3x}$$

Auxiliary equation is $D^2+6D+9 = 0$ or $D = -3, -3.$

$$\text{C.F.} = (C_1+C_2x)e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2+6D+9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2+6(3)+9} = \frac{5e^{3x}}{36}$$

The complete solution is $y = (C_1+C_2x)e^{-3x}+\frac{5e^{3x}}{36}$

Example 2. Solve

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y = 6e^{3x}$$

Solution:

$$(D^2-6D+9)y = 6e^{3x}$$

A.E. is $(D^2-6D+9) = 0$ or $(D-3)^2 = 0$, or $D = 3, 3$

$$\text{C.F.} = (C_1+C_2x)e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2-6D+9}6e^{3x} = x\frac{1}{2D-6}6e^{3x} = x^2\frac{1}{2}\cdot 6\cdot e^{3x} = 3x^2e^{3x}$$

Complete solution is $y = (C_1+C_2x)e^{3x}+3x^2e^{3x}$

Case II. When X = sin(ax + b) or cos(ax + b).

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b) \text{ provided } f(-a^2)\neq 0 \quad \ldots(iv)$$

If $f(-a^2) = 0$, the above rule fails and we can prove that,

$$\therefore \quad \frac{1}{f(D^2)}\cdot\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b) \text{ provided } f'(-a^2)\neq 0 \quad \ldots(v)$$

If $f'(-a^2) = 0$, $\frac{1}{f(D^2)}\cdot\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$, provided $f''(-a^2)\neq 0$ and so on...

Similarly, $\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$, provided $f(-a^2)\neq 0$.

If $f(-a^2) = 0$, $\frac{1}{f(D^2)}\cos(ax+b) = x\cdot\frac{1}{f'(-a^2)}\cos(ax+b)$, provided $f'(-a^2)\neq 0$.

If $f'(-a^2) = 0$, $\frac{1}{f(D^2)}\cos(ax+b) = x^2\cdot\frac{1}{f''(-a^2)}\cos(ax+b)$ provided $f''(-a^2)\neq 0$ and so on...

ILLUSTRATIVE EXAMPLES

Example 1. Solve

$$(D^2 + 4)y = \sin 3x.$$

Solution:

$$(D^2 + 4)y = \sin 3x$$

Auxiliary equation is

$$D^2 + 4 = 0 \text{ or } D = \pm 2i.$$

$$\text{C.F.} = A\cos 2x + B\sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cdot\sin 3x = \frac{\sin 3x}{(-3)^2+4} = \frac{1}{5}\sin 3x$$

Complete solution is

$$y = A\cos 2x + B\sin 2x - \frac{1}{5}\sin 3x$$

Example 2. Solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos 2x.$$

Solution:

$$(D^2 + D + 1)y = \cos 2x$$

Auxiliary equation is

$$D^2 + D + 1 = 0$$

$$D = \frac{-1 \pm \sqrt{-3}}{2}, \text{ C.F.} = e^{-\frac{x}{2}}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}x}{2}\right]$$

$$\text{P.I.} = \frac{1}{D^2+D+1}\cdot\cos 2x$$

$$= \frac{1}{(-2^2)+D+1}\cdot\cos 2x = \frac{1}{D-3}\cdot\cos 2x$$

$$= \frac{D+3}{D^2-9}\cdot\cos 2x = \frac{D+3}{(-2^2)-9}\cos 2x$$

$$= -\frac{1}{13}(D+3)\cos 2x = -\frac{1}{13}(-2\sin 2x + 3\cos 2x)$$

Complete solution is

$$y = e^{-x/2}\left[A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right] + \frac{1}{13}[2\sin 2x - 3\cos 2x]$$

Example 3. Solve

$$(D^2 + 4)y = \cos 2x$$

Solution:

$$(D^2 + 4)y = \cos 2x$$

Auxiliary equation is

$$D^2 + 4 = 0$$

$$D = \pm 2i, \text{ C.F.} = A\cos 2x + B\sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cos 2x = x\cdot\frac{1}{2D}\cos 2x = \frac{x}{2}\left(\frac{1}{2}\sin 2x\right) = \frac{x}{4}\sin 2x$$

Complete solution is

$$y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$$

Case III. When $X = x^m$.

Here $$\text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the (m + 1)th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

ILLUSTRATIVE EXAMPLES

Example 1. Solve

Find the P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

Solution:

Given equation in symbolic form is $(D^2 + D)y = x^2 + 2x + 4$.

$$\therefore \quad \text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^2-\ldots)(x^2+2x+4) = \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \int(x^2+4)dx = \frac{x^3}{3}+4x.$$

Case IV. When $X = e^{ax}V$, V being a function of x.

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D + a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D + a)^2u$$

and in general, $$D^n(e^{ax}u) = e^{ax}(D + a)^n u$$

$$\therefore \quad f(D)(e^{ax}u) = e^{ax}f(D + a)u$$

Operating both sides by 1/f (D),

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

Now put $$f(D + a)u = V,$$

i.e. $$u = \frac{1}{f(D+a)}V,$$

so that $$e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}(e^{ax}V)$$

i.e. $$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$$

ILLUSTRATIVE EXAMPLES

Example 1. Solve

$$(D^2 - 4D + 4)y = x^3e^{2x}$$

Solution:

$$(D^2 - 4D + 4)y = x^3 e^{2x}$$

A.E. $$D^2 - 4D + 4 = 0, (D-2)^2 = 0 \text{ or } D = 2, 2$$

$$\text{C.F.} = (C_1 + C_2 x)e^{2x}$$

$$\text{PI} = \frac{1}{D^2 - 4D + 4} x^3 \cdot e^{2x} = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x} \frac{1}{D^2} x^3 = e^{2x} \cdot \frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x} \cdot \frac{x^5}{20}$$

$$y = (C_1 + C_2 x)\, e^{2x} + e^{2x} \cdot \frac{x^5}{20}$$

Example 2. Solve

$$(D^2 - 5D + 6)y = e^x \cos 2x$$

Solution:

$$(D^2 - 5D + 6)y = e^x \cos 2x$$

$$D^2 - 5D + 6 = 0$$

$$(D-2).(D-3) = 0, \text{ or } D = 2, 3$$

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^x \cos 2x$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 5(D+1) + 6} \cos 2x$$

$$= e^x \cdot \frac{1}{D^2 - 3D + 2} \cos 2x = e^x \cdot \frac{1}{-4 - 3D + 2} \cos 2x$$

$$= -e^x \frac{1}{3D + 2} \cos 2x = -e^x \frac{3D - 2}{9D^2 - 4} \cos 2x$$

$$= -e^x \frac{3D - 2}{9(-4) - 4} \cos 2x = \frac{e^x}{40}(3D - 2)\cos 2x$$

$$= \frac{e^x}{40}(-6 \sin 2x - 2 \cos 2x) = -\frac{e^x}{20}(3 \sin 2x + \cos 2x)$$

$$y = C_1 e^2 + C_2 e^{3x} - \frac{e^x}{20}(3 \sin 2x + \cos 2x)$$

Case V. When X is any other function of x.

Here $$\text{P.I.} = \frac{1}{f(D)} X.$$

If $f(D) = (D - m_1)(D - m_2) \ldots D(D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}$$

$\therefore$ $$\text{P.I.} = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right] X$$

$$= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \ldots + A_n \frac{1}{D - m_n} X$$

$$= A_1 \cdot e^{m_1x}\int Xe^{-m_1x}dx + A_2 \cdot e^{m_2x}\int Xe^{-m_2x}dx + \ldots + A_n \cdot e^{m_nx}\int Xe^{-m_nx}dx$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.57 For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$

(c) $3e^{2x}$ (d) $C_1e^{-x} + C_2e^{-3x}$

[ME, GATE-2006, 2 marks]

Solution: (b)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$$

$$\Rightarrow \quad (D^2 + 4D + 3)y = 3e^{2x}$$

Particular integral $\quad P.I. = \frac{1}{D^2 + 4D + 3}3e^{2x}$

Now since, $\quad \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

$$P.I. = 3\frac{e^{2x}}{(2)^2 + 4(2) + 3} = \frac{3e^{2x}}{15} = \frac{e^{2x}}{5}$$

Q.58 Consider two solutions $x(t) = x_1(t)$ and $x(t) = x_2(t)$ of the differential equation $\frac{d^2x(t)}{dt^2} + x(t) = 0$, $t > 0$, such that $x_2 = 0, \left.\frac{dx_2(t)}{dt}\right|_{t=0} = 1$. The Wronskian $W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{vmatrix}$ at $t = \pi/2$ is

(a) 1 (b) −1

(c) 0 (d) $\pi/2$

[ME, GATE-2014 : 2 Marks, Set-3]

Solution : (a)

Given differential equation in symbolic form is $(D^2 + 1)\,x(t) = 0$

Its A.E. is $D^2 + 1 = 0$,

$\therefore \quad D = \pm i$

So, C.F. is

$$x_1(t) = C_1 \cos t,$$

$$x_2(t) = C_2 \sin t$$

$\therefore \quad x_1(0) = C_1 = 1$

$\Rightarrow \quad x_1(t) = \cos t \quad \left[\text{it satisfies } \left.\frac{dx_1(t)}{dt}\right|_{t=0} = 0\right]$

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$$x_2(0) = 0 = C_2 \sin(0) \qquad (\because C_2 \neq 0)$$

$$\frac{dx_2(t)}{dt} = C_2 \cos t$$

$$\Rightarrow \quad \left.\frac{dx_2(t)}{dt}\right|_{t=0} = C_2 = 1$$

$$\therefore \quad x_2(t) = \sin t$$

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$W(t) = \cos^2 t + \sin^2 t = 1$$

Q.59 Find the solution of $\frac{d^2y}{dx^2} = y$ which passes through the origin and the point $\left(ln\, 2, \frac{3}{4}\right)$.

(a) $y = \frac{1}{2}e^x - e^{-x}$

(b) $y = \frac{1}{2}\left(e^x + e^{-x}\right)$

(c) $y = \frac{1}{2}\left(e^x - e^{-x}\right)$

(d) $y = \frac{1}{2}e^x + e^{-x}$

[ME, GATE-2015 : 2 Marks, Set-1]

Solution: (c)

$$\frac{d^2y}{dx^2} = y$$

$$\Rightarrow \quad D^2y = y \qquad (\therefore d/dx = D)$$

$$(D^2 - 1)y = 0$$

$$D^2 - 1 = 0$$

$$D = \pm 1$$

$$y = C_1 e^x + C_2 e^{-x}$$

Given point passes through origin

$$\Rightarrow \quad O = C_1 + C_2$$

$$C_1 = -C_2 \qquad \text{...(i)}$$

Also, point passes through (ln 2, 3/4)

$$\Rightarrow \quad \frac{3}{4} = C_1 e^{\ln 2} + C_2 e^{-\ln 2}$$

$$\frac{3}{4} = 2C_1 + \frac{C_2}{2}$$

$$\Rightarrow \quad C_2 + 4C_1 = 1.5 \qquad \text{...(ii)}$$

From (i) $\quad C_1 = -C_2$, putting in (ii), we get

$$\Rightarrow \quad -3C_2 = 1.5$$

$$C_2 = -0.5$$

$$\therefore \quad C_1 = 0.5$$

$$\Rightarrow \quad y = 0.5\,(e^x - e^{-x})$$

$$y = \frac{e^x - e^{-x}}{2}$$

Q.60 If the characteristic equation of the differential equation $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$ has two equal roots, then the values of α are

(a) ± 1
(b) 0, 0
(c) $\pm j$
(d) $\pm 1/2$

[EC, GATE-2014 : 1 Mark, Set-2]

Solution : (a)

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$$

The characteristic equation is given as

$$(m^2 + 2\alpha m + 1) = 0$$

$$m_1, m_2 = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4}}{2}$$

Since both roots are equal i.e.

$$m_1 = m_2$$

$$\frac{-2\alpha + \sqrt{4\alpha^2 - 4}}{2} = \frac{-2\alpha - \sqrt{4\alpha^2 - 4}}{2}$$

$$\sqrt{4\alpha^2 - 4} = -\sqrt{4\alpha^2 - 4}$$

$$2\sqrt{4\alpha^2 - 4} = 0$$

$$4\alpha^2 - 4 = 0$$

$$\alpha^2 = 1$$

$$\alpha = \pm 1$$

Q.61 If a and b are constants, the most general solution of the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$ is

(a) ae^{-t}
(b) $ae^{-t} + bte^{-t}$
(c) $ae^{t} + bte^{-t}$
(d) ae^{-2t}

[EC, GATE-2014 : 1 Mark, Set-4]

Solution : (b)

The differential equation is given as

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

$$y = C \cdot F + P \cdot I$$

Since Q = 0, i.e. RHS term is zero, so there will be no particular integral.

$$\therefore \quad y = C \cdot F$$

Let $$\frac{\partial}{\partial x} = D$$

So, $(D^2 + 2D + 1)x = 0$

$$\therefore \quad (D + 1)^2 = 0$$

$$\therefore \quad y = ae^{-t} + bte^{-t}$$

Q.62 The particular solution of the initial value problem given below is

$$\frac{d^2y}{dx^2}+12\frac{dy}{dx}+36y=0 \text{ with } y(0)=3 \text{ and } \left.\frac{dy}{dx}\right|_{x=0}=-36$$

(a) $(3-18x)e^{-6x}$
(b) $(3+25x)e^{-6x}$
(c) $(3+20x)e^{-6x}$
(d) $(3-12x)e^{-6x}$

[EC, 2016 : 2 Marks, Set-3]

Solution: (a)

$$(D^2+12D+36)^4 = 0$$
$$(D+6)^2y = 0$$
$$D = -6, -6$$
$$y = (C_1 + xC_2)e^{-6x}$$
$$y = C_1e^{-6x} + C_2xe^{-6x}$$
$$y(0) = 33 = C_1 + 0$$
$\Rightarrow$ $$C_1 = 3$$
$$y' = -6C_1e^{-6x} + C_2e^{-6x} - 6C_2xe^{-6x}$$
$$y'(0) = -36$$
$$-36 = -6C_1 + C_2$$
$$-36 = -18 + C_2$$
$$C_2 = -18$$
$\therefore$ $$y = 3e^{-6x} - 18xe^{-6x}$$
$$y = (3-18x)e^{-6x}$$

Q.63 If $y=f(x)$ satisfies the boundary value problem $y''+9y=0$, $y(0)=0$, $y\left(\frac{\pi}{2}\right)=\sqrt{2}$, then $y\left(\frac{\pi}{4}\right)$ is ___.

[ME, 2016 : 2 Marks, Set-1]

Solution:

$$(D^2+9)y = 0$$
$$m^2+9 = 0$$
$$m = \pm 3i$$
$$y = C_1\cos 3x + C_2\sin 3x \quad \text{...(i)}$$
$x = 0$ $$\theta = C_1$$
$$y\left(\frac{\pi}{2}\right) = \sqrt{2}$$
$$\sqrt{2} = C_1\cos\frac{3\pi}{2} + C_2\sin\frac{3\pi}{2}$$
$$C_2 = -\sqrt{2}$$
$\therefore$ $$y = \sqrt{2}\sin 3x$$
$$y\left(\frac{\pi}{4}\right) = -\sqrt{2}\sin\frac{3\pi}{4} = \frac{-5}{4} = -1$$

Q.64 The respective expressions for complimentary function and particular integral part of the solution of the differential equation

$$\frac{d^4y}{dx^4}+3\frac{d^2y}{dx^2}=108x^2 \text{ are}$$

(a) $\left[c_1 + c_2 x + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(b) $\left[c_2 + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$

(c) $\left[c_1 + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(d) $\left[c_1 + c_2 x + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$ [CE, 2016 : 2 Marks, Set-I]

Solution: (a)

D.E. is $(D^4 + 3D^2)\, y = 108x^2$, $D = \dfrac{d}{dx}$

A.E. $m^4 + 3m^2 = 0$

$\Rightarrow \quad m^2(m^2 + 3) = 0$

$\Rightarrow \quad m = 0, 0, \pm\pm\sqrt{3}\, i$

$\therefore$ CF $= (C_1 + C_2 x) + C_3 \sin\left(\sqrt{3}x\right) + C_4 \cos\left(\sqrt{3}\,x\right)$

and PI $= \dfrac{1}{D^4 + 3D^2}\left(108x^2\right) = \dfrac{1}{3D^2\left[1 + \dfrac{D^2}{3}\right]}\left(108x^2\right) = \dfrac{36}{D^2}\left[1 + \dfrac{D^2}{3}\right]^{-1}\left(x^2\right)$

$= \dfrac{36}{D^2}\left[1 - \dfrac{D^2}{3} + \dots\right]\left(x^2\right) = \dfrac{36}{D^2}\left[x^2 - \dfrac{1}{3}(2) + 0\right] = \iint\left(36x^2 - \dfrac{2}{3}\right)dx\,dx$

$= 36\left(\dfrac{x^4}{(4)(3)} - \dfrac{2}{3}\dfrac{x^2}{(2)(1)}\right) = 3x^4 - 12x^2$

3.3.5 Summary: Working Procedure to Solve The Equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

of which the symbolic form is

$$(D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = X.$$

Step 1. To Find the Complementary Function

1. Write the A.E.
   i.e. $D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n = 0$ and
2. Write the C.F. as follows

Roots of A.E.	C.F.
1. $m_1, m_2, m_3 \dots$ (real and different roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots$
2. $m_1, m_1, m_3 \dots$ (two real and equal roots)	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots$
3. $m_1, m_1, m_1, m_4 \dots$ (three real and equal roots)	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots$
4. $\alpha + i\beta, \alpha - i\beta, m_3 \dots$ (a pair of imaginary roots)	$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \dots$ (2 pairs of equal imaginary roots)	$e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right] + c_5 e^{m_5 x} + \dots$

Step II. To Find the Particular Integral

From symbolic form

$$\text{P.I.} = \frac{1}{D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n} X$$

$$= \frac{1}{f(D)} \text{ or } \frac{1}{\phi(D^2)} X.$$

1. When $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $f'(D)$ = diff. coeff. of $f(D)$ w.r.t. D

$f''(D)$ = diff. coeff. of $f'(D)$ w.r.t. D, etc.

2. When $X = \sin(ax + b)$ or $\cos(ax + b)$.

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $D^2 = -a^2$ $\qquad [\phi(-a^2) \neq 0]$

$$= x \frac{1}{\phi'(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $D^2 = -a^2$ $\qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $D^2 = -a^2$ $\qquad [\phi'(-a^2) = 0, \phi''(-a^2) \neq 0.$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D)^2$ w.r.t. D, etc.

3. When $X = x^m$, m being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m.$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

4. When $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D + a)} V$$

and then evaluate $\frac{1}{f(D + a)} V$ as in (i), (ii), and (iii).

5. When X is any function of x.

$$\text{P.I.} = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx$$

Setp III. To find the complete solution:

Then the C.S. is y = C.F. + P.I.

3.4 TWO OTHER METHODS OF FINDING P.I.

3.4.1 Method of Variation of Parameters

This method is quite general and applies to equations of the from

$$y'' + py' + qy = X \quad \text{... (i)}$$

where p, q, and X are functions of x. It gives

$$\text{P.I.} = -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx \quad \text{... (ii)}$$

where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ ... (iii)

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

ILLUSTRATIVE EXAMPLES

Example: 1

Using the method of variation of parameters, solve

$$y'' + y = \sec x$$

Solution:

Given equation in symbolic form is $(D^2 + 1)y = \sec x$.

(a) To find C.F.

Its A.E. is $D^2 + 1 = 0$,

$\therefore$ $D = \pm i$

Thus C.F. is $y = c_1 \cos x + c_2 \sin x$

(b) To find P.I.

Here $y_1 = \cos x$, $y_2 = \sin x$ and $X = \sec x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Thus,

$$\text{P.I.} = -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$$

$$= -\cos x\int\frac{\sin x \sec x\, dx}{1} + \sin x\int\frac{\cos x \sec x\, dx}{1}$$

$$= -\cos x\int \tan x\, dx + \sin x\int 1.dx$$

$$= \cos x \ln \cos x + x \sin x$$

Hence the C.S. is

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln \cos x + x \sin x$$

$$= (c_1 + \ln \cos x) \cos x + (c_2 + x) \sin x$$

3.5 EQUATIONS REDUCIBLE TO LINEAR EQUATION WITH CONSTANT COEFFICIENT

Definitions

Now, we shall study linear differential equation with variable coefficients, which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

Euler-Cauchy differential equation.

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = Q(x)$$

It can be reduced into linear differential equations with constant coefficients.

By taking $x = e^t$ (or) $t = \log x$

Let, $$\theta = \frac{d}{dt}$$

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$\Rightarrow$ $$x\frac{dy}{dx} = \theta y$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dt}\right] = -\frac{1}{x^2}\frac{dx}{dt} + \frac{1}{x}\frac{d}{dt}\left[\frac{dy}{dt}\right]\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$$

Similarly, $$x^3\frac{d^3y}{dx^3} = \theta(\theta - 1)(\theta - 2)y \text{ and so on.}$$

Substitute all these values in given differential equation, it results in a linear equation with constant coefficients. Which can be solved as above methods.

Example: 1

Consider the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$

with the boundary conditions of $y(0) = 0$, $y(1) = 1$, the complete solution of the differential equation is

(a) x^2

(b) $\sin\frac{\pi x}{2}$

(c) $e^x \sin\frac{\pi x}{2}$

(d) $e^{-x} \sin\frac{\pi x}{2}$

Solution: (a)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0 \quad \text{and} \quad y(0) = 0,\ y(1) = 1$$

Choice (a) satisfies the initial condition as well as equation as shown in below

if $$y = x^2$$

$\Rightarrow$ $$y(0) = 0,\quad y(1) = 1^2 = 1$$

Substitution in differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y = x^2\times 2+x\times 2x-4x^2=0$$

$\therefore$ $4 = x^2$ is complete solution

Alternate solution:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y = 0$$
$$(x^2D^2+xD-4)y = 0$$
$$[\theta(\theta-1)+\theta-4]y = 0$$
$$(\theta^2-\theta+\theta-4) = 0$$
$$(\theta^2-4)y = 0$$

Auxilliary equation is $m^2-4=0$

$$m = \pm 2$$

CF is $C_1e^{-2z}+C_2e^{2z}$

Solution is $y = C_1e^{-2z}+C_2e^{2z}=C_1x^{-2}+C_2x^2=C_1\frac{1}{x^2}+C_2x^2$

One of the independent solution is x^2.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.65 Consider the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$. Which of the following is a solution to this differential equation for $x>0$?

(a) e^x
(b) x^2
(c) $1/x$
(d) $ln\,x$ **[EE, GATE-2014 : 1 Mark, Set-2]**

Solution : (c)

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y = 0$$

Let $x = e^z \longleftrightarrow z=\log x$

$$x\frac{d}{dx} = xD=\theta=\frac{d}{dz}$$
$$x^2D^2 = \theta(\theta-1)$$
$$(x^2D^2+xD-1)y = 0$$
$$[\theta(\theta-1)+\theta-1]y = 0$$
$$(\theta^2-\theta+\theta-1) = 0$$
$$(\theta^2-1)y = 0$$

Auxilliary equation is $m^2-1=0$

$$m = \pm 1$$

CF is $C_1e^{-z}+C_2e^{z}$

Solution is $y = C_1e^{-z}+C_2e^{z}=C_1x^{-1}+C_2x=C_1\frac{1}{x}+C_2x$

One independent solution is $\frac{1}{x}$

Another independent solution is x.

OOOO

CHAPTER 4

Complex Functions

4.1 INTRODUCTION

Many engineering problems may be treated and solved by methods involving complex numbers and complex functions. There are two kinds of such problems. The first of them consists of "elementary problems" for which some acquaintance with complex numbers is sufficient. This includes many applications to electric circuits or mechanical vibrating systems.

The second kind consists of more advanced problems for which we must be familiar with the theory of complex analytic functions— **"complex function theory"** or **"complex analysis,"** for short— and with its powerful and elegant methods. Interesting problems in heat conduction, fluid flow, and electrostatics belong to this category.

We shall see that the importance of complex analytic functions in engineering mathematics has the following two main roots.

1. The real and imaginary parts of an analytic function are solutions of Laplace's equation in two independent variables. Consequently, two-dimensional potential problems can be treated by methods developed for analytic functions.
2. Most higher functions is engineering mathematics are analytic functions, and their study for complex values of the independent variable leads to a much deeper understanding of their properties. Furthermore, complex integration can help evaluating complicated complex and real integrals of practical interest.

4.2 COMPLEX FUNCTIONS

If for each value of the complex variable z (= $x + iy$) in a given region R, we have one or more values of w (= $u + iv$), then w is said to be a complex function of z and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y.

If to each value of z, there corresponds one and only one value of w, then w is said to be a single-valued function of z otherwise a multi-valued function. For example $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

4.2.1 Exponential Function of a Complex Variable

When x is real, we are already familiar with the exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \infty.$$

Similarly, we define the exponential function of the complex variable $z = x + iy$, as

$$e^z \text{ or } \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots\infty \quad \ldots(i)$$

Putting $x = 0$ in (i), we get, $z = iy$ and

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots\infty$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)$$

$$= \cos y + i \sin y$$

Thus $\quad e^z = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

Also $\quad x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore$ Exponential form of $z\ (= x + iy) = re^{i\theta}$.

4.2.2 Circular Function of a Complex Variable

Since, $\quad e^{iy} = \cos y + i \sin y$

and $\quad e^{-iy} = \cos y - i \sin y$

$\therefore$ The circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to define the circular functions of the complex variable z by the equations:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}$$

with cosec z, sec z and cot z as their respective reciprocals.

Cor. 1. Euler's Theorem. By definition

$$\cos z + i \sin z = \frac{e^{iz} - e^{-iz}}{2i} + i\frac{e^{iz} - e^{-iz}}{2i} = e^{iz} \quad \text{where } z = x + iy$$

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \cos\theta + i\sin\theta$, where θ is real or complex. This is called the Euler's theorem.*

Cor. 2. De Moivre's theorem for complex numbers.

Whether θ is real or complex, we have

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

4.2.3 Hyperbolic Functions

1. **Def.** If x be real or complex,

(a) $\dfrac{e^x - e^{-x}}{2}$ is defined as hyperbolic sine of x and is written as sinh x.

(b) $\dfrac{e^x + e^{-x}}{2}$ is defined as hyperbolic cosine of x and is written as cosh x.

Thus, $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Also we define, $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$; $\coth x = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$$\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \ \text{cosech } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Cor. $\sinh 0 = 0$, $\cosh 0 = 1$ and $\tanh 0 = 0$.

2. Relations between hyperbolic and circular functions.

   Since for all values of θ, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

   ∴ Putting θ = ix, we have

$$\sin ix = \frac{e^{-x} - e^{x}}{2i} = -\left[\frac{e^x - e^{-x}}{2i}\right] \qquad [\because e^{i\theta} = e^{i \cdot ix} = e^{-x}]$$

$$= i^2 \frac{e^x - e^{-x}}{2i} = i \cdot \frac{e^x - e^{-x}}{2} = i \sinh x$$

and $\cos ix = \dfrac{e^{-x} + e^{x}}{2} = \cosh x$

Thus,

$$\sin ix = i \sinh x \qquad \text{... (i)}$$
$$\cos ix = \cosh x \qquad \text{... (ii)}$$

and ∴

$$\tan ix = i \tanh x \qquad \text{... (iii)}$$

Cor.

$$\sinh ix = i \sin x \qquad \text{... (iv)}$$
$$\cosh ix = \cos x \qquad \text{... (v)}$$
$$\tanh ix = i \tan x \qquad \text{... (vi)}$$

4.2.4 Inverse Hyperbolic Functions

Def. If $\sinh u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

4.2.5 Logarithmic Function of a Complex Variable

1. **Def.** If z(= x + iy) and w(= u + iv) be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$. ... (i)

   Also $e^{w + 2in\pi} = e^w \cdot e^{2in\pi} = z$ $\qquad [\because e^{2in\pi} = 1]$

   ∴ $\log z = w + 2in\pi$ ... (ii)

   i.e. the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function. The general value of the logarithm of z is written as Log z (beginning with capital L) so as to distinguish it from its principal value which is written as log z. This principal value is obtained by taking n = 0 in Log z.

   Thus from (i) and (ii), $\text{Log}(x + iy) = 2in\pi + \log(x + iy)$.

   Obs.

   (a) If y = 0, then $\text{Log } x = 2in\pi + \log x$.

   This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

(b) We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.

e.g.
$$\begin{aligned} \log_e(-2) &= \log_e 2(-1) \\ &= \log_e 2 + \log_e(-1) \\ &= \log_e 2 + i\pi \qquad [\because -1 = \cos\pi + i\sin\pi = e^{i\pi}] \\ &= 0.6931 + i(3.1416) \end{aligned}$$

2. Real and imaginary parts of Log(x + iy).

$$\begin{aligned} \text{Log}(x+iy) &= 2in\pi + \log(x+iy) \qquad \text{Put, } x = r\cos\theta,\ y = r\sin\theta \\ &= 2in\pi + \log[r(\cos\theta + i\sin\theta)] \qquad \text{so that } r = \sqrt{(x^2+y^2)} \\ &= 2in\pi + \log(re^{i\theta}) \qquad \text{and } \theta = \tan^{-1}(y/x) \\ &= 2in\pi + \log r + i\theta \\ &= \log\sqrt{(x^2+y^2)} + i[2n\pi + \tan^{-1}(y/x)] \end{aligned}$$

3. Real and imaginary parts of $(a + i\beta)^{x+iy}$

$$(a + i\beta)^{x+iy} = e^{(x+iy)\,\text{Log}(\alpha+i\beta)} \qquad \left[\begin{array}{l}\text{Put } \alpha = r\cos\theta,\ \beta = r\sin\theta \text{ so that} \\ r = \sqrt{(\alpha^2+\beta^2)} \text{ and } \theta = \tan^{-1}\beta/\alpha.\end{array}\right.$$

$$\begin{aligned} &= e^{(x+iy)[2in\pi + \log(\alpha+i\beta)]} \\ &= e^{(x+iy)[2n\pi + \log re^{i\theta}]} \\ &= e^{(x+iy)[\log r + i(2n\pi+\theta)]} \\ &= e^{A+iB} \\ &= e^{A}(\cos B + i\sin B) \end{aligned}$$

where $A = x\log r - y(2n\pi + \theta)$ and $B = y\log r + x(2n\pi + \theta)$.

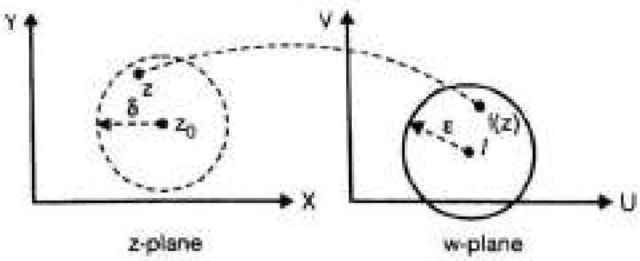
4.3 LIMIT OF A COMPLEX FUNCTION

A function $w = f(z)$ is said to tend to limit l as z approaches a point z_0, if for real ϵ, we can find a positive real δ such that

$$|f(z) - l| < \epsilon \text{ for } |z - z_0| < \delta$$

i.e. for every $z \neq z_0$ in the δ-disc (dotted) of z-plane, $f(z)$ has a value lying in the ϵ-disc of w-plane (see figure below). In symbols, we write $\underset{z \to z_0}{\text{Lt}} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different, for in real calculus x approaches x_0 only along the line whereas here z approaches z_0 from any direction in the z-plane.



Continuity of f(z). A function $w = f(z)$ is said to be **continuous** at $z = z_0$, if

$$\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$$

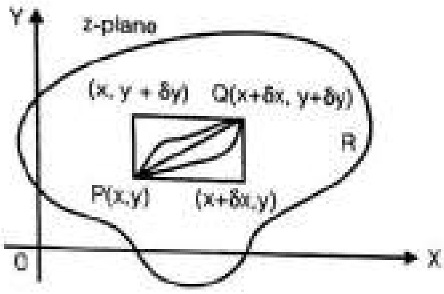
Further f(z) is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, i.e. at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then f(z) will be continuous at $z = z_0$.

4.4 DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then the derivative of $w = f(z)$ is defined to be

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z+\delta z) - f(z)}{\delta z},$$



provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Suppose P(z) is fixed and Q(z + δz) is a neighbouring point (Figure above). The point Q may approach P along any straight or curved path in the given region, i.e. δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

Theorem. The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R, are

1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R;
2. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$

The relations in (ii) are known as Cauchy-Riemann* equations or briefly C-R equations.

4.5 ANALYTIC FUNCTIONS

A function f(z) which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic** or a **regular function** of z in that region.

A point at which an analytic function ceases to possess a derivative is called a **singular point** of the function.

Thus if u and v are real single-valued functions of x and y such that ∂u/∂x, ∂u/∂y, ∂v/∂x, ∂v/∂y are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \dots (i)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of f(z) is then, given by

$$f'(z) = \underset{\delta x \to 0}{Lt}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \quad \text{...(ii)}$$

or

$$f'(z) = \underset{\delta y \to 0}{Lt}\left(\frac{\partial u}{i\delta y} + i\frac{\partial v}{i\delta y}\right)$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y \quad \text{...(iii)}$$

The real and imaginary parts of an analytic function are called **conjugate functions**. The relation between two conjugate functions is given by the C-R equations (i) above.

ILLUSTRATIVE EXAMPLES

Example 1:

Is $f(z) = z^3$ analytic?

Solution:

$$z = x + iy$$

$\Rightarrow$ $z^2 = (x + iy)(x + iy) = x^2 - y^2 + 2ixy$

$\Rightarrow$ $z^3 = (x^2 - y^2 + 2ixy)(x + iy)$

$$= (x^3 - 3xy^2) + (3x^2 y - y^3)i$$

here $u = x^3 - 3xy^2$

$$v = 3x^2y - y^3$$

$$u_x = 3x^2 - 3y^2, \quad v_y = 3x^2 - 3y^2$$

$$u_y = -6xy, \quad v_x = 6xy$$

So $u_x = v_y$ and $u_y = -v_x$

So C-R equations are satisfied and also the partial derivatives are continuous at all points. Hence z^3 is analytic for every z.

Example 2:

If $w = \log z$, find dw/dz and determine where w is non-analytic.

Solution:

We have $w = u + iv = \log(x + iy)$

$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$$

so that $u = \frac{1}{2}\log(x^2 + y^2)$, $v = \tan^{-1} y/x$.

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at z = 0.

$\therefore$ $$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)}$$

$$= \frac{1}{x + iy} = \frac{1}{z} (z \neq 0).$$

Obs. The definition of the derivates of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. Thus if, **a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 For an analytic function, $f(x + iy) = u(x, y) + iv(x, y)$, u is given by $u = 3x^2 - 3y^2$. The expression for v, considering K to be a constant is

(a) $3y^2 - 3x^2 + K$
(b) $6x - 6y + K$
(c) $6y - 6x + K$
(d) $6xy + K$

[CE, GATE-2011, 2 mark]

Solution: (d)

$$f = u + iv$$
$$u = 3x^2 - 3y^2$$

for f to be analysis, we have Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(i)$$
$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \quad ...(ii)$$

From (i) we have

$$6x = \frac{\partial v}{\partial y}$$
$$\Rightarrow \quad \int \partial v = \int 6x \partial y$$
$$v = 6\frac{x^2}{2} + f(x)$$

i.e.,
$$v = 3x^2 + f(x) \quad ...(iii)$$

Now applying equation (ii) we get

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$
$$\Rightarrow \quad -6y = -\left[6x + \frac{df}{dx}\right]$$
$$\Rightarrow \quad 6x + \frac{df}{dx} = 6y$$
$$\frac{df}{dx} = 6y - 6x$$

By integrating,

$$f(x) = 6yx - 3x^2 + K$$

Substitute in equation (iii)

$$v = 3x^2 + 6yx - 3x^2 + K$$
$$\Rightarrow \quad v = 6yx + K$$

Q.2 $z = \dfrac{2 - 3i}{-5 + i}$ can be expressed as

(a) $-0.5 - 0.5i$
(b) $-0.5 + 0.5i$
(c) $0.5 - 0.5i$
(d) $0.5 + 0.5i$

[CE, GATE-2014 : 1 Mark, Set-2]

Solution : (b)

$$\frac{(2-3i)}{(-5+i)} = \frac{(2-3i)}{(-5+i)} \times \frac{(-5-i)}{(-5-i)} = \frac{-10-2i+15i-3}{25+1} = \frac{-13+13i}{26} = -0.5 + 0.5i$$

Q.3 An analytic function of a complex variable z = x + iy is expressed as f(z) = u (x, y) + i v(x, y) where i = $\sqrt{-1}$. If u = xy, the expression for v should be

(a) $\frac{(x+y)^2}{2}+k$ (b) $\frac{x^2-y^2}{2}+k$

(c) $\frac{y^2-x^2}{2}+k$ (d) $\frac{(x-y)^2}{2}+k$

[ME, GATE-2009, 2 marks]

Solution: (c)

$$f(z) = u + iv \text{ is analytic (given)}$$

∴ It must satisfy the Cauchy-Reimann equations

$$u_x = v_y \quad \text{...(i)}$$

and $$v_x = -u_y \quad \text{...(ii)}$$

Here since, $$u = xy \text{ (given)}$$

⇒ $$u_x = y \text{ and } u_y = x$$

Now substituting u_x and u_y (i) and (ii) we get

$$v_y = y \quad \text{...(iii)}$$

and $$v_x = -x \quad \text{...(iv)}$$

Integrating (iii) and (iv) we can now get v as follows:

$$v_y = y$$

⇒ $$\frac{\partial v}{\partial y} = y$$

⇒ $$\int \partial v = \int y\, \partial y$$

⇒ $$v = \frac{y^2}{2} + f(x) \quad \text{...(v)}$$

from (v) we have, $$v_x = f'(x) \quad \text{...(vi)}$$

Since from (iv) we have, $$v_x = -x$$

Substituting this is (vi) we get,

$$f'(x) = -x$$

⇒ $$\frac{df}{dx} = -x$$

⇒ $$\int df = \int -x\, dx$$

⇒ $$f = \frac{-x^2}{2} + k$$

Now substitute this is (v) we get,

$$v = \frac{y^2}{2} - \frac{x^2}{2} + k \quad ; \quad v = \frac{y^2-x^2}{2} + k$$

Q.4 The modulus of the complex number $\left(\frac{3+4i}{1-2i}\right)$ is

(a) 5
(b) $\sqrt{5}$
(c) $1/\sqrt{5}$
(d) 1/5

[ME, GATE-2010, 1 mark]

Solution: (b)

$$Z = \frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{-5+10i}{5} = -1+2i$$

$$|z| = \sqrt{(-1)^2+(2)^2} = \sqrt{5}$$

Q.5 An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + i\,v(x, y)$, where $i = \sqrt{-1}$. If $u(x, y) = 2xy$, then $v(x, y)$ must be

(a) $x^2 + y^2$ + constant
(b) $x^2 - y^2$ + constant
(c) $-x^2 + y^2$ + constant
(d) $-x^2 - y^2$ + constant

[ME, GATE-2014 : 2 Marks, Set-2]

Solution : (c)

As per Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2y$$

and $$\frac{\partial u}{\partial y} = 2x$$

$$\frac{\partial v}{\partial y} = 2y$$

$\Rightarrow$ $$v = y^2 + f(x)$$

$$\frac{\partial v}{\partial x} = 0 + f'(x) = -2x$$

$\therefore$ $$f(x) = -x^2 + \text{constant}$$

$\therefore$ $$v = y^2 - x^2 + \text{constant}$$

Q.6 An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + iv(x, y)$, where $i = \sqrt{-1}$. If $u(x, y) = x^2 - y^2$, then expression for $v(x, y)$ in terms of x, y and a general constant c would be

(a) $xy + c$
(b) $\frac{x^2+y^2}{2} + c$
(c) $2xy + c$
(d) $\frac{(x-y)^2}{2} + c$

[ME, GATE-2014 : 2 Marks, Set-3]

Solution : (c)

As per Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\therefore \quad \frac{\partial v}{\partial x} = 2x$$

$$\Rightarrow \quad v = 2xy + f(x)$$

$$\frac{\partial v}{\partial x} = 2y + f'(x)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = 2y + f'(x)$$

$$\Rightarrow \quad f'(x) = 0 \quad \text{i.e. } f(x) = C$$

$$\therefore \quad v = 2xy + C$$

Q.7 The argument of the complex number $\frac{1+i}{1-i}$, where $i = \sqrt{-1}$, is

(a) $-\pi$ (b) $-\frac{\pi}{2}$

(c) $\frac{\pi}{2}$ (d) π

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (c)

Let $$z = \frac{1+i}{1-i}$$

or $$z = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i^2+2i}{1-i^2} = \frac{2i}{2} = i$$

$$z = x + iy = i$$

so, $$x = 0$$

$$y = 1$$

$$\text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

Q.8 Given two complex numbers $z_1 = 5 + (5\sqrt{3})i$ and $z_2 = \frac{2}{\sqrt{3}} + 2i$ the argument of $\frac{z_1}{z_2}$ in degree is

(a) 0 (b) 30

(c) 60 (d) 90

[ME, GATE-2015 : 1 Mark, Set-1]

Solution: (a)

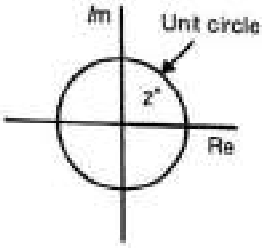
$$z_1 = 5+\left(5\sqrt{3}\right)i \quad ; \quad z_2 = \frac{2}{\sqrt{3}}+2i$$

$$\arg(z_1) = \theta_1 = \tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) \quad ; \quad \theta_1 = 60°$$

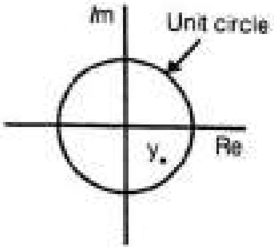
$$\arg(z_2) = \theta_2 = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) \quad ; \quad \theta_2 = 60°$$

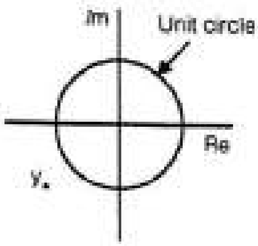
$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = 60 - 60 = 0°$$

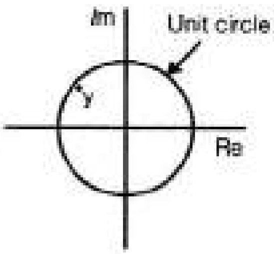
Q.9 A point z has been plotted in the complex plane, as shown in figure below.

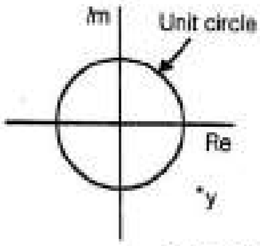


The plot for point $\frac{1}{z}$ is

(a)


(b)


(c)


(d)


[EE, GATE-2011, 1 marks]

Solution: (d)

Let $Z = a + bi$

Since Z is shown inside the unit circle in I quadrant, a & b are both +ve and $0<\sqrt{a^2+b^2<1}$

Now $\frac{1}{Z} = \frac{1}{a+bi}$

$$\frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Since a, b > 0,

$$\frac{a}{\sqrt{a^2+b^2}} > 0$$

$$\frac{-b}{a^2+b^2} < 0$$

So $\frac{1}{Z}$ is in IV quadrant.

$$\left|\frac{1}{Z}\right| = \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{-b}{a^2+b^2}\right)^2} = \sqrt{\frac{1}{a^2+b^2}} = \frac{1}{\sqrt{a^2+b^2}}$$

Since $$0 < \sqrt{a^2+b^2} < 1$$

$$\frac{1}{\sqrt{a^2+b^2}} > 1$$

So $\frac{1}{Z}$ is outside the unit circle is IV quadrant.

Q.10 If $x = \sqrt{-1}$, then the value of x^x is

(a) $e^{-\pi/2}$
(b) $e^{\pi/2}$
(c) x
(d) 1

[EC, EE, IN, GATE-2012, 1 mark]

Solution. (a)

$$x = i, \text{ then in polar coordinates,}$$

$$x = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{\pi}{2}i}$$

Now, $$x^x = i^i = \left(e^{\pi i/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2}$$

Q.11 Square roots of $-i$, where $i = \sqrt{-1}$, are

(a) $i, -i$
(b) $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$
(c) $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$
(d) $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right), \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

[EE, GATE-2013, 1 Mark]

Solution: (b)

$$-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$$

$$(-i)^{1/2} = \left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]^{1/2}$$

$$= \cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

Q.12 Let S be the set of points in the complex plane corresponding to the unit circle. (That is, $S = \{z : |z| = 1\}$). Consider the function $f(z) = zz^*$ where z^* denotes the complex conjugate of z. The $f(z)$ maps S to which one of the following in the complex plane

(a) unit circle
(b) horizontal axis line segment from origin to (1, 0)
(c) the point (1, 0)
(d) the entire horizontal axis

[EE, GATE-2014 : 1 Mark, Set-1]

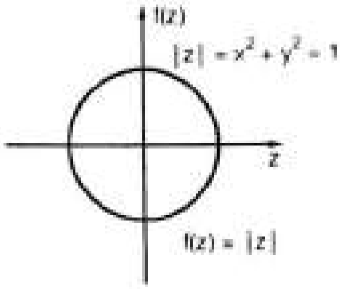
Solution : (c)

zz^*

$$\Rightarrow \quad z = x + iy$$

$$z^* = x - iy$$

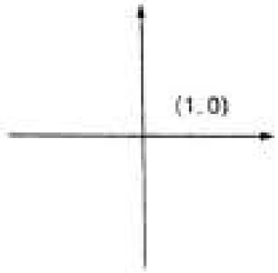
$$zz^* = (x + iy)(x - iy) = x^2 + y^2$$



which is equal to (1) always as given

$$|z| = 1$$

$$zz^* = x^2 + y^2$$



Q.13 All the values of the multi-valued complex function 1^i, where $i = \sqrt{-1}$, are

(a) purely imaginary
(b) real and non-negative
(c) on the unit circle
(d) equal in real and imaginary parts

[EE, GATE-2014 : 1 Mark, Set-2]

Solution : (b)

Let $$z = 1^i = 1^{e^{i(4n+1)\frac{\pi}{2}}} \quad n \in I$$

$z = 1$ which is purely real and non negative.

Q.14 Given $f(z) = g(z) + h(z)$, where f, g, h are complex valued functions of a complex variable z. Which one of the following statements is TRUE?

(a) If $f(z)$ is differentiable at z_0, then $g(z)$ and $h(z)$ are also differentiable at z_0.
(b) If $g(z)$ and $h(z)$ are differentiable at z_0, then $f(z)$ is also differentiable at z_0.
(c) If $f(z)$ is continuous at z_0, then it is differentiable at z_0.
(d) If $f(z)$ is differentiable at z_0, then so are its real and imaginary parts.

[EE, GATE-2015 : 1 Mark, Set-2]

Answer: (b)

Q.15 The complex function tanh(s) is analytic over a region of the imaginary axis of the complex s-plane if the following is **TRUE** everywhere in the region for all integers n

(a) $\text{Re}(s) = 0$
(b) $\text{Im}(s) \neq n\pi$
(c) $\text{Im}(s) \neq \frac{n\pi}{3}$
(d) $\text{Im}(s) \neq \frac{(2n+1)\pi}{2}$

[IN, GATE-2013 : 1 mark]

Solution: (d)

$$\tanh s = \frac{e^s - e^{-s}}{e^s + e^{-s}}$$

it is analytic if $e^s + e^{-s} \neq 0$

$$\therefore \quad e^s \neq e^{-s}$$
$$e^{2s} \neq -1$$
$$s \neq \frac{i(2n+1)}{2}\pi$$
$$\therefore \quad \text{Im}(s) \neq \frac{(2n+1)}{2}\pi$$

Q.16 The real part of an analytic function f(z) where z = x + jy is given by $e^{-y}\cos(x)$. The imaginary part of f(z) is

(a) $e^y\cos(x)$ (b) $e^{-y}\sin(x)$

(c) $-e^y\sin(x)$ (d) $-e^{-y}\sin(x)$

[EC, GATE-2014 : 2 Marks, Set-2]

Answer: (d)

Q.17 Let $z = x + iy$ be a complex variable. Consider that contour integration is performed along the unit circle in anticlockwise direction. Which one of the following statements is NOT TRUE?

(a) The residue of $\frac{z}{z^2 - 1}$ at $z = 1$ is 1/2

(b) $\oint_C z^2 dz = 0$

(c) $\frac{1}{2\pi i}\oint_C \frac{1}{z} dz = 1$

(d) $\bar{z}$ (complex conjugate of z) is analytical function

[EC, GATE-2015 : 1 Mark, Set-1]

Solution: (d)

$$f(z) = \bar{z} = x - iy$$
$$u = x \qquad v = -y$$
$$\Rightarrow \quad u_x = 1 \qquad v_x = 0$$
$$u_y = 0 \qquad v_y = -1$$
$$u_x \neq v_y \text{ i.e. } C - R \text{ not satisfied}$$

$\Rightarrow$ $\bar{z}$ is not analytic function.

Q.18 Let $f(z) = \frac{az + b}{cz + d}$. If $f(z_1) = f(z_2)$ for all $z_1 \neq z_2$, $a = 2$, $b = 4$ and $c = 5$, then d should be equal to ____.

[EC, GATE-2015 : 1 Mark, Set-2]

Solution: (10)

$$f(z_1) = \frac{az_1 + b}{cz_1 + d}$$

$$f(z_2) = \frac{az_2 + b}{cz_2 + d}$$

$$\frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d}$$

$$acz_1z_2 + bcz_2 + adz_1 + bd = acz_1z_2 + bcz_1 + adz_2 + bd$$

$$bc(z_2 - z_1) = ad(z_2 - z_1)$$

$$z_2 \neq z_1$$

$\Rightarrow$ $$bc = ad$$

$$d = \frac{bc}{a} = \frac{4 \times 5}{2} = 10$$

Q.19 In the neighborhood of $z = 1$, the function $f(z)$ has a power series expansion of the form
$f(z) = 1 + (1 - z) + (1 - z)^2 + \ldots\ldots\ldots$
Then $f(z)$ is

(a) $\frac{1}{z}$ (b) $\frac{-1}{z-2}$

(c) $\frac{z-1}{z+2}$ (d) $\frac{1}{2z-1}$

[IN, 2016 : 1 Mark]

Solution: (a)

$$f(z) = 1 + (1-z) + (1-z)^2 + ---$$

$$= \frac{1}{1-(1-z)} = \frac{1}{1-1+z} = \frac{1}{z}$$

Q.20 Consider the complex valued function
$f(z) = 2z^3 + b|z|^3$ where z is a complex variable. The value of b for which the function $f(z)$ is analytic is _____

[EC, 2016 : 1 Mark, Set-2]

Solution:

$$f(z) = 2z^3 + b_1|z|^3$$

Given that $f(z)$ is analytic.
which is possible only when $b = 0$

since $|z^3|$ is differentiable at the origin but not analytic.
$2z^3$ is analytic everywhere

$\therefore$ $$f(z) = 2z^3 + b|z^3| \text{ is analytic}$$

only when $$b = 0$$

Q.21 $f(z) = u(x, y) + iv(x, y)$ is an analytic function or complex variable $z = x + iy$ where $i = \sqrt{-1}$,
$u(x, y) = 2xy$, then $v(x, y)$ may be expressed as

(a) $-x^2 + y^2$ + constant (b) $x^2 - y^2$ + constant

(c) $x^2 + y^2$ + constant (d) $-(x^2 + y^2)$ + constant

[ME, 2016 : 1 Mark, Set-1]

Solution: (a)

$$u = 2xy$$
$$u_x = 2y \quad u_y = 2x$$

In option (a)

$$V_x = -2x \quad u_y = -V_x$$
$$V_y = 2y$$

(–R equation are satisfied only in option a)

Q.22 A function f of the complex variable $z = x + iy$, is given as $f(x, y) = u(x, y) + iv(x, y)$, where $u(x, y) = 2kxy$ and $v(x, y) = x^2 - y^2$. The value of k, for which the function is analytic, is ____

[ME, 2016 : 1 Mark, Set-2]

Solution:

Given that $f(z) = u + iv$ is analytic

$$u(x, y) = 2kxy \qquad v = x^2 - y^2$$
$$u_x = 2ky \qquad v_y = -2y$$
$$u_x = v_y$$
$$k = -1$$
$$u_y = 2kx \qquad v_x = 2x$$
$$u_y = -v_x$$
$$2kx = -2x$$
$$k = -1$$

Q.23 Consider the function $f(z) = z + z^*$ where z is a complex variable and z^* denotes its complex conjugate. Which one of the following is TRUE?

(a) $f(z)$ is both continuous and analytic
(b) $f(z)$ is continuous but not analytic
(c) $f(z)$ is not continuous but is analytic
(d) $f(z)$ is neither continuous nor analytic

[EE, 2016 : 1 Mark, Set-2]

Solution: (b)

$$f(z) = z + z^*$$
$$f(z) = 2x \text{ is continuous (polynomial)}$$
$$u = 2x \qquad v = 0$$
$$u_x = 2 \qquad u_y = 0$$
$$v_x = 0 \qquad v_y = 0$$

C.R. equiation not satisfied.

∴ No where analytic.

4.6 COMPLEX INTEGRATION

4.6.1 Line integral in the complex plane

As in calculus we distinguish between definite integrals and indefinite integrals or antiderivatives. An **indefinite integral** is a function whose derivative equals a given analytic function in a region. By known differentiation formulas we may find many types of indefinite integrals.

Complex definite integrals are called (complex) **line integrals.** They are written as

$$\int_C f(z)dz$$

Here the **integrand** f(z) is integrated over a given curve C in the complex plane, called the **path of integration**. We may represent such a curve C by a parametric representation.

(1) $\boxed{x(t) = x(t) + iy(t)}$ $(a \le t \le b)$.

The sense of increasing t is called the **positive sense** on C, and we say that in this way, (1) **orients** C.
We assume C to be a **smooth curve**, that is, C has a continuous and nonzero derivative $\dot{z} = dz/dt$ at each point. Geometrically this means that C has a unique and continuously turning tangent.

Definition of the complex line integral

This is similar to the method in calculus. Let C be a smooth curve in the complex plane given by (1), and let f(z) be a continuous function given (at least) at each point of C. We now subdivide (we "**partition**") the interval $a \le t \le b$ in (1) by points

$$t_0 (=a),\ t_1,\ \dots,\ t_{n-1},\ t_n\ (= b)$$

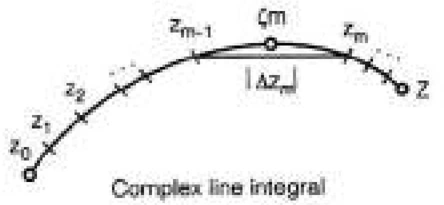
where $t_0 < t_1 \cdots < t_n$. To this subdivision there corresponds a subdivision of C by points

$$z_0,\ z_1,\ \cdots,\ z_{n-1},\ z_n\ (= Z)$$

where $z_j = z(t_j)$. On each portion of subdivision of C we choose an arbitrary point, say, a point ζ_1 between z_0 and z_1 (that is, $\zeta_1 = z(t)$) where t satisfies $t_0 \le t \le t_1$), a point ζ_2 between z_1 and z_2, etc. Then we form the sum

(2) $$S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m \quad \text{where} \quad \Delta z_m = z_m - z_{m-1}.$$

We do this for each n = 2, 3, ⋯ in a completely independent manner, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$. This implies that the greatest $|\Delta z_m|$ also approaches zero because it cannot exceed the length of the arc of C from z_{m-1} to z_m and the latter goes to zero since the arc length of the smooth curve C is a continuous function of t. The limit of the sequence of complex numbers $S_2, S_3, \cdots$ thus obtained is called the **line Integral** (or simply the integral) of f(z) over the oriented curve C. This



Complex line integral

curve C is called **path of integration**. The line integral is denoted by

(3) $\boxed{\int_C f(z)dz,}$ or by $\boxed{\oint_C f(z)dz}$

if C is a **closed path** (one whose terminal point Z coincides with its initial point z_0, as for a circle or an 8-shaped curve).

General Assumption. All paths of integration for complex line integrals are assumed to be **piecewise smooth**, that is, they consist of finitely many smooth curves joined end to end.

First method: indefinite integration and substitution of limits

This method is simpler than the next one, but is less general. It is restricted to analytic functions. Its formula (9) (below) is the analog of the familiar formula from calculus

$$\int_a^b f(x)dx = F(b) - F(a) \qquad [F'(x) = f(x)].$$

Theorem 1: (Indefinite integration of analytic functions)

Let f(z) be analytic in a simply connected domain D. A domain D is called **simply connected** if every simple closed curve (closed curve without self-intersections in D encloses only points of D). Then there exists an indefinite integral of f(z) in the domain D, that is, an analytic function F(z) such that F'(z) = f(z) in D, and for all paths in D joining two points z_0 and z_1 in D we have

(4) $$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

This theorem will be proved in the next section.

Simple connectedness is quite essential in Theorem 1, as we shall see in Example 5. Since analytic functions are our main concern, and since differentiation formulas will often help in finding F(z) for a given f(z) = F'(z), the present method is of great practical interest.

If f(z) is entire, we can take for D the complex plane (which is certainly simply connected).

ILLUSTRATIVE EXAMPLES

Example 1: $$\int_0^{1+i} z^2 dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3}+\frac{2}{3}i$$

Example 2: $$\int_{-\pi i}^{\pi i} \cos z\, dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i\sinh \pi = 23.097i$$

Example 3: $$\int_{8+\pi i}^{8-3\pi i} e^{z/2}dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$$

Since e^z is periodic with period $2\pi i$.

Example 4: $$\int_{-i}^{i}\frac{dz}{z} = \text{Ln}\, i - \text{Ln}(-i) = \frac{i\pi}{2} - \left(\frac{i\pi}{2}\right) = i\pi.$$

Here D is the complex plane without 0 and the negative real axis (where Ln z is not analytic), obviously a simply connected domain.

Second method: use of a representation of the path

This method is not restricted to analytic functions but applies to any continuous complex function.

Theorem 2: (Integration by the use of the path)

Let C be a piecewise smooth path, represented by z = z(t), where $a \le t \le b$. Let f(z) be a continuous function on C. Then

(5) $$\int_C f(z)dz = \int_a^b f[z(t)]\,\dot{z}(t)dt \qquad \left(\dot{z} = \frac{dz}{dt}\right)$$

Proof: The left side of (5) is given in terms of real line integrals as $\int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$.

We now show that the right side of (5) also equals the same.

We have z = x + iy, hence $\dot{z} = \dot{x} + i\dot{y}$. We simply write u for u[x(t), y(t)] and v for v[x(t), y(t)]. We also have $dx = \dot{x}\, dt$ and $dy = \dot{y}\, dt$.

Consequently, in (5)

$$\int_a^b f[z(t)\dot{z}(t)]dt = \int_a^b (u+iv)(\dot{x}+i\dot{y})dt = \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)]$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx).$$

Steps in applying Theorem 2

(A) Represent the path C in the form z(t) ($a \le t \le b$).

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute z(t) for every z in f(z) (hence x(t) for x and y(t) for y).

(D) Integrate $f[z(t)]\ \dot{z}(t)$ over t from a to b.

ILLUSTRATIVE EXAMPLES

Example 1: A basic result: Integral of 1/z around the unit circle

We show that by integrating 1/z counterclockwise around the unit circle (the circle of radius 1 and center 0), we obtain

(6) $$\boxed{\oint_C \frac{dz}{z} = 2\pi i}$$ (C the unit circle, counterclockwise).

This is a very important result that we shall need quite often.

Solution: We may represent the unit circle C in the form

$$z(t) = \cos t + i \sin t = e^{it} \qquad (0 \le t \le 2\pi),$$

so that the counterclockwise integration corresponds to an increase of t from 0 to 2π. By differentiation, $\dot{z}(t) = ie^{it}$ (chain rule) and with $f(z(t)) = 1/z(t) = e^{-it}$ we get from (10) the result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it} dt = i\int_0^{2\pi} dt = 2\pi i.$$

Check this result by using z(t) = cos t + i sin t.

Simple connectedness is essential in Theorem 1. Equation (4) in Theorem 1 gives 0 for any closed path because then $z_1 = z_0$, so that $F(z_1) - F(z_0) = 0$. Now 1/z is not analytic at z = 0. But any simply connected domain containing the unit circle must contain z = 0, so that Theorem 1 does not apply—it is not enough that 1/z is analytic in an annulus, say $\frac{1}{2} < |z| < \frac{3}{2}$, because an annulus is not simply connected!

Example 2: Integral of integer powers

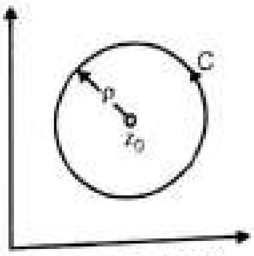
Let $f(z) = (z - z_0)^m$ where m is an integer and z_0 a constant. Integrate counterclockwise around the circle C of radius ρ with center at z_0 (Fig. below)

Solution: We may represent C in the form

$$z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \qquad (0 \le i \le 2\pi).$$

Then we have

$$(z - z_0)^m = \rho^m e^{imt}, \qquad dz = i\rho e^{it}\,dt$$



Path in Example 2

and obtain

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it} dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt.$$

By the Euler formula, the right side equals

$$i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t\, dt + i\int_0^{2\pi} \sin(m+1)t\, dt\right].$$

If $m = -1$, we have $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. We thus obtain $2\pi i$. For integer $m \neq 1$ each of the two integrals is zero because we integrate over an interval of length 2π, equal to a period of sine cosine. Hence the result is

$$\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m=-1), \\ 0 & (m \neq -1 \text{ and integer}). \end{cases} \quad (7)$$

Dependence on path. Now comes a very important fact. If we integrate a given function f(z) from a point z_0 to a point z_1 along different paths, the integrals will in general have different values. In other words, **a complex line integral depends not only on the endpoints of the path but in general also on the path itself.** See the next example.

Example 3: Integral of a nonanalytic function. Dependence on path

Integrate $f(z) = \text{Re } z = x$ from 0 to $1 + 2i$

(a) along C^* in Fig. below,

(b) along C consisting of C_1 and C_2.

Solution:

(a) C^* can be represented by $z(t) = t + 2it$ $(0 \le t \le 1)$. Hence $\dot{z}(t) = 1 + 2i$ and $f[f(t)] = x(t) = t$ on C^*.

We now calculate

$$\int_{C^*} \text{Re } z\, dz = \int_0^1 t(1+2i)dt = \frac{1}{2}(1+2i) = \frac{1}{2} + i.$$

(b) We now have

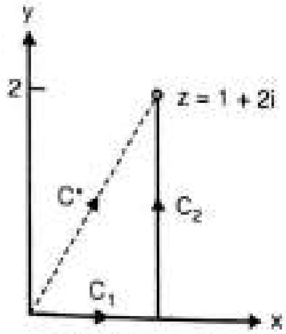
$$C_1: z(t) = t, \quad \dot{z}(t) = 1, \quad f(z(t)) = x(t) = t \quad (0 \le t \le 1)$$

$$C_2: z(t) = 1 + it, \quad \dot{z}(t) = i, \quad f(z(t)) = x(t) = t \quad (0 \le t \le 2)$$

We calculate by partitioning the path C into two paths C_1 and C_2 as shown below

$$\int_C \text{Re } z\, dz = \int_{C_1} \text{Re } z\, dz + \int_{C_2} \text{Re } z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \frac{1}{2} + 2i.$$

Note that this result differs from the result in (a).



Paths in Example 3

4.7 CAUCHY'S THEOREM

If f(z) is an analytic function and f'(z) is continuous at each point within and on a closed curve C, then $\int_C f(z)dz = 0$.

Writing f(z) = u(x, y) + iv(x, y) and noting that dz = dx + idy

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \quad \text{... (i)}$$

Since f'(z) is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence the Green's theorem can be applied to (i), giving

$$\int_C f(z)dz = -\iint_C \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]dx\,dy + i\iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]dx\,dy \quad \text{... (ii)}$$

Now f(z) being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (ii) vanish identically.

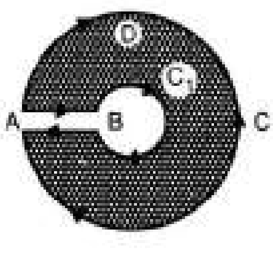
Hence, $\iint_C f(z)dz = 0.$

Obs. 1 The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. Hence the line integral of a function f(z) which is analytic in the region D, is independent of the path joining any two points of D.

Obs. 2 Extension of Cauchy's theorem. If f(z) is analytic in the region D between two simple closed curves C and C_1, then $\int_C f(z)dz = \int_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\int f(z)dz = 0$ where the path is as indicated by arrows in Figure below i.e. along AB—along C_1 in clockwise sense & along BA—along C in anti-clockwise sense

i.e. $\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz + \int_C f(z)dz = 0.$



But, since the integral along AB and along BA cancel, it follows that

$$\int_C f(z)dx + \int_{C_1} f(z)dz = 0.$$

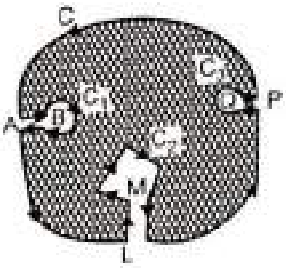
Reversing the direction of the integral around C_1 and transposing, we get

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

each integration being taken in the anti-clockwise sense.

If $C_1, C_2, C_3, \ldots$, be any number of closed curves within C (Figure below), then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \ldots$$



4.8 CAUCHY'S INTEGRAL FORMULA

If f(z) is analytic within and on a closed curve and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}.$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz \qquad \begin{cases}\text{For any point on } C_1, \\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}d\theta\end{cases}$$

$$= \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta}d\theta = i\int_{C_1} f(a+re^{i\theta})d\theta. \qquad \ldots (i)$$

In the limiting form, as the circle C_1 shrinks to the point a, i.e. as $r \to 0$, the integral (i) will approach to

$$i\int_{C_1} f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a). \text{ Thus } \int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

i.e.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \qquad \ldots (ii)$$

which is the desired Cauchy's integral formula.

Cor. Differentiating both sides of (2) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz. \qquad \ldots (iii)$$

Similarly,

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz \qquad \ldots (iv)$$

and in general,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \qquad \ldots (v)$$

Thus it follows from the results (2) to (5) that if a function f(z) is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C. Incidently, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic**.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.24 Using Cauchy's integral theorem, the value of the integral (integration being taken in counterclockwise direction) $\oint_c \frac{z^3-6}{3z-i}dz$ is

(a) $\frac{2\pi}{81} - 4\pi i$
(b) $\frac{\pi}{8} - 6\pi i$
(c) $\frac{4\pi}{81} - 6\pi i$
(d) 1

[CE, GATE-2006, 2 marks]

Solution: (a)

Cauchy's integral theorem is

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz$$

i.e.

$$\oint_c \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

Now,

$$\oint_c \frac{z^3-6}{3z-i}dz = \frac{1}{3}\oint_c \frac{z^3-6}{\left(z-\frac{i}{3}\right)}$$

Applying Cauchy's integral theorem, using $f(z) = z^3 - 6$,

$$= \frac{1}{3}\left(2\pi i\, f\left(\frac{i}{3}\right)\right) = \frac{1}{3}\left(2\pi i\left[\left(\frac{i}{3}\right)^3 - 6\right]\right) = \frac{1}{3}\left[2\pi i\left(\frac{i^3}{27} - 6\right)\right] = \frac{2\pi}{81}i^4 - 4\pi i = \frac{2\pi}{81} - 4\pi i$$

Q.25 The value of the integral $\int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}dz$ (where C is a closed curve given by $|z| = 1$) is

(a) $-\pi i$
(b) $\frac{\pi i}{5}$
(c) $\frac{2\pi i}{5}$
(d) πi

[CE, GATE-2009, 2 marks]

Soltuon: (c)

Here,

$$I = \int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}dz = \frac{1}{2}\int_C \frac{\left[\frac{\cos(2\pi z)}{(z-3)}\right]}{\left[z-\frac{1}{2}\right]}$$

Since, $z = 1/2$ is a point within $|z| = 1$ (the closed curve C) we can use Cauchy's integral theorem and say that

$$I = \frac{1}{2}f\left(\frac{1}{2}\right)$$

where

$$f(z) = \frac{\cos(2\pi z)}{(z-3)}$$

[Notice that f(z) is analytic on all pts inside $|z| = 1$]

$\therefore$

$$I = \frac{1}{2}\frac{\cos\left(2\pi \times \frac{1}{2}\right)}{\left(\frac{1}{2}-3\right)} = \frac{2\pi i}{5}$$

Q.26 If z is a complex variable, the value of $\int_5^{3i} \frac{dz}{z}$ is

(a) $-0.511 - 1.57i$ (b) $-0.511 + 1.57i$
(c) $0.511 - 1.57i$ (d) $0.511 + 1.57i$

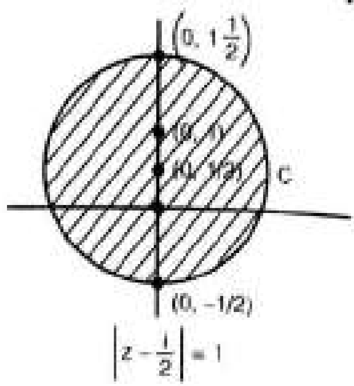
[ME, 2014 : 2 Marks, Set-2]

Answer: (b)

Q.27 The value of $\oint_C \frac{dz}{(1+z^2)}$ where C is the contour $|z-i/2| = 1$ is

(a) $2\pi i$ (b) π
(c) $\tan^{-1} z$ (d) $\pi i \tan^{-1} z$ [EE, GATE-2007, 2 marks]

Solution: (b)



$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

Poles at i and –i, i.e. (0, 1) and (0, –1)
From figure of $|z - i/2| = 1$ below we see that pole (0, 1) i.e. i is inside C, while pole (0, –1) i.e. –i is outside C.

So, $$I = 2\pi i \text{ Res } f(i) = 2\pi i \cdot \frac{1}{(i-i)(i+i)} = \pi$$

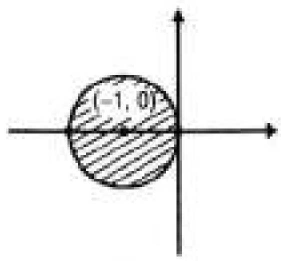
Q.28 Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in the z-plane such that $|z + 1| = 1$, the value of $\frac{1}{2\pi j}\oint_C f(z)dz$ is

(a) -2 (b) -1
(c) 1 (d) 2 [EC, EE, IN, GATE-2012, 1 mark]

Solution: (c)

Given, $$f(z) = \frac{1}{z+1} - \frac{2}{z+3} = \frac{(z+3)-2(z+1)}{(z+1)(z+3)} = \frac{-z+1}{(z+1)(z+3)}$$

Poles are at –1 and –3 i.e. (–1, 0) and (–3, 0).
From figure below of $|z + 1| = 1$,
we see that (–1, 0) is inside the circle and (–3, 0) is outside the circle.



Residue theorem says,

$$\frac{1}{2\pi j}\oint_C f(z)\,dz = \text{Residue of those poles which are inside C.}$$

So the required integral $\frac{1}{2\pi j}\oint_C f(z)dz$ is given by the residue of function at pole (–1, 0) (which is inside the circle).

This residue is $= \frac{-(-1)+1}{(-1+3)} = \frac{2}{2} = 1$

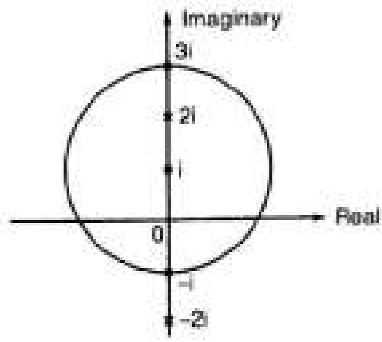
Q.29 $\oint \frac{z^2-4}{z^2+4} dz$ evaluated anticlockwise around the circle $|z-i| = 2$, where $i = \sqrt{-1}$, is

(a) -4π
(b) 0
(c) $2+\pi$
(d) $2+2i$

[EE, GATE-2013, 2 Marks]

Solution: (a)

$$\frac{Z^2-4}{Z^2+4} = \frac{Z^2-4}{(Z+2i)(Z-2i)}$$



Poles at 2i and −2i i.e. (0, 2i) and (0, −2i)

From figure of $|Z-i| = 2$, we see that pole, is incide C,

While pole, −2i is outside C.

$$\therefore \quad \oint \frac{Z^2-4}{Z^2+4} dz = 2\pi i \times \text{Res. } F(z)$$

$$= 2\pi i \cdot \left. \frac{(Z-2i)(Z^2-4)}{(Z+2i)(Z-2i)} \right|_{Z=2i} = 2\pi i \frac{\left[(2i)^2-4\right]}{(2i+2i)} = -4\pi$$

Q.30 Integration of the complex function $f(z) = \frac{z^2}{z^2-1}$, in the counterclockwise direction, around $|z-1| = 1$, is

(a) $-\pi i$
(b) 0
(c) πi
(d) $2\pi i$

[EE, GATE-2014 : 2 Marks, Set-3]

Solution : (c)

$$f(z) = \int \frac{z^2}{z^2-1} = \int f(z)$$

Given circle

$$|z-1| = 1$$

$$\Rightarrow \quad |(x+iy)-1| = 1$$

$$(x-1)^2 + y^2 = 1$$

$$x = 1, y = 0, r = 1$$

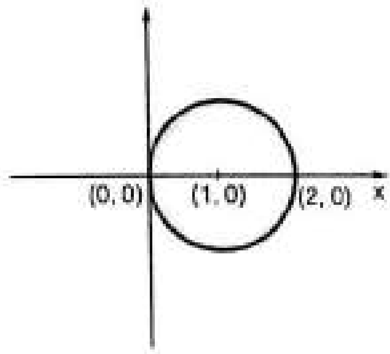
Poles of f(z)

$$z^2 - 1 = 0$$

$$[z = +1, 1]$$

So, $-1 \rightarrow$ Outside circle

$+1 \rightarrow$ Inside circle



$$\int \frac{z^2}{(z-1)(z+1)} = 2\pi i \left[\frac{z^2}{z+1}\right]_{z=+1} = 2\pi i \left[\frac{1}{2}\right] = \pi i$$

For pole $(z=-1) = \int \frac{z^2}{(z-1)(z+1)} = 0$

as it lies outside from counter.

Q.31 The value of $\oint \frac{1}{z^2}dz$, where the contour is the unit circle traversed clockwise, is

(a) $-2\pi i$ (b) 0
(c) $2\pi i$ (d) $4\pi i$ [IN, GATE-2015 : 1 Mark]

Solution: (b)

Given, $\oint \frac{1}{z^2}dz$ where C is the unit circle. By Cauchy's residue theorem

$$\oint \frac{1}{z^2}dz = 2\pi i \text{ [sum of residues]}$$

$\because$ $\frac{1}{z^2}$ is NOT analytical at $z = 0$

So, $z = 0$ is the pole of order 2.

So, residue at $z = 0 = \frac{1}{\underline{2-1}}\left[\frac{d}{dz}z^2.\frac{1}{z^2}\right]_{z=0}$

$$= 0$$

So, $$\oint \frac{1}{z^2}dz = 2\pi i[0] = 0$$

Q.32 If C denotes the counterclockwise unit circle, the value of the contour integral $\frac{1}{2\pi j}\oint_c \text{Re}\{z\}dz$ is ____. [EC, GATE-2015 : 2 Marks, Set-2]

Solution: (1/2)

C: $x^2 + y^2 = 1$

$z = x + iy$; $dz = dx + idy$

$$x = \cos\theta \;;\; y = \sin\theta \;;\; 0 \le \theta \le 2\pi$$
$$dx = -\sin\theta d\theta \;;\; dy = \cos\theta d\theta$$

$$\frac{1}{2\pi i}\oint_c x(dx + idy) = \frac{1}{2\pi i}\int_0^{2\pi}\cos\theta(-\sin\theta d\theta) + i\cos\theta\cos\theta d\theta$$

$$= \frac{1}{2\pi i}\left[\int_0^{2\pi} -\sin\theta\cos\theta d\theta + i\int_0^{2\pi}\cos^2\theta d\theta\right]$$

$$= \frac{1}{2\pi i}\left[\int_0^{2\pi} -\frac{\sin 2\theta}{2}d\theta + i\int_0^{2\pi}\left[\frac{1+\cos 2\theta}{2}\right]d\theta\right]$$

$$= \frac{1}{2\pi i}\left[\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} + i\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi i}\left[\left(\frac{1}{2} - \frac{1}{2}\right) + i[\pi]\right] = \frac{1}{2}$$

Q.33 If C is a circle of radius r with centre z_0, in the complex z-plane and if n is a non-zero integer, then $\oint \frac{dz}{(z-z_0)^{n+1}}$ equals

(a) $2\pi nj$ (b) 0
(c) $\frac{\pi j}{2\pi}$ (d) $2\pi n$ [EC, GATE-2015 : 1 Mark, Set-3]

Solution: (b)

By Cauchy integral formula

$$\oint \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i\, f^n(z_0)}{n!}$$

$$\oint \frac{dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!}\cdot 0 = 0$$

Q.34 The value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}dx$$

evaluated using contour integration and the residue theorem is

(a) $\frac{-\pi \sin(1)}{e}$

(b) $\frac{-\pi \cos(1)}{e}$

(c) $\frac{\sin(1)}{e}$

(d) $\frac{\cos(1)}{e}$

[ME, 2016 : 2 Marks, Set-1]

Solution: (a)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}dx$$

$$I = \int_{-\infty}^{\infty} \frac{\sin z}{z^2+2z+2}dz = \int_{-\infty}^{\infty} \frac{\text{I.P of } e^{iz}}{z^2+2z+2}dz$$

Poles are $z^2+2z+2=0$

$$z = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm 2i}{2} = -1\pm i$$

$z = -1-i$

↓

Outside upper half

↓

Residue is 0

$-1+i$

↓

inside upperhalf

$$\underset{z=-1+i}{\text{Res }\phi(z)} = \underset{z\to -1+i}{Lt}\; z-(-1+i)\times \frac{e^{iz}}{(z-(-1+i)\,z-(-1-9))}$$

$$= \frac{e^{i(-1+i)}}{(-1+i)-(-1-i)} = \frac{e^{-i-1}}{-1+i+1+i} = \frac{e^{-i-1}}{2i}$$

$$I = \text{I.P. of } 2\pi i\left(\frac{e^{-i-1}}{2i}\right) = \text{I.P of } \pi\left(e^{-i}\cdot e^{-1}\right)$$

$$= \text{I.P of } \pi e^{-1}(\cos 1^\circ - i\sin 1^\circ) = \frac{-\pi \sin 1^\circ}{e}$$

Q.35 The value of the integral

$$\oint \frac{2z+5}{\left(z-\frac{1}{2}\right)(z^2-4z+5)}dz$$

over the contour $|z| = 1$, taken in the anti-clockwise direction, would be

(a) $\frac{24\pi i}{13}$

(b) $\frac{48\pi i}{13}$

(c) $\frac{24}{13}$

(d) $\frac{12}{13}$

[EE, 2016 : 1 Mark, Set-1]

Solution: (b)

Singlarilies, $Z = \frac{1}{2}, 2 \pm i$

only, $Z = \frac{1}{2}$ lies inside C

By residue theorem,

$$\oint_C = 2\pi i(R) = \frac{48\pi i}{13}$$

Residue at $\frac{1}{2}$ $= R_{1/2} = \lim_{Z\to 1/2}\left[\left(Z-\frac{1}{2}\right)\cdot\frac{2Z+5}{\left(Z-\frac{1}{2}\right)(Z^2+4Z+5)}\right] = \frac{24}{13}$

4.9 SERIES OF COMPLEX TERMS

1. **Taylor's series***. If f(z) is analytic inside a circle C with centre at a, then for z inside C.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots\ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots\ldots \quad \ldots (i)$$

2. **Laurent's series***. If f(z) is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a, then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a\_1(z-a)^{-1} + a_{-2}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt$$

Γ being any curve in R, encircling C_1 (as in Figure below).

Obs. 1. As f(z) is analytic inside, G, then $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt \neq \frac{f^n(a)}{n!}$

However, if f(z) is analytic inside G, then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt = \frac{f^n(a)}{n!}$

and Laurent's series reduces to Taylor's series.

Obs. 2. To obtain Taylor's or Laurent's series, simply expand f(z) by binomial theorem, instead of finding a_n by complex integration which is quite complicated.

Obs. 3. Laurent series of a given analytic function f(z) in its annulus of convergence is unique. There may be different Laurent series of f(z) in two annuli with the same centre.

4.10 ZEROS AND SINGULARITES OR POLES OF AN ANALYTIC FUNCTION

4.10.1 Zeros of an Analytic Function

Def. A zero of an analytic function f(z) is that value of z for which f(z) = 0

If f(z) is analytic in the neighbourhood of a point z = a, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_n(z-a)^n + \ldots \text{ where } a_n = \frac{f^n(a)}{n!}$$

If $a_0 = a_1 = a_2 = \ldots = a_{m-1} = 0$ but $a_m \neq 0$, then f(z) is said to have a zero of order m at z = a.

When m = 1, the zero is said to be simple. In the neighbourhood of zero (z = a) of order m.

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \ldots\infty$$
$$= (z-a)^m \phi(z)$$

where, $\phi(z) = a_m + a_{m+1}(z-a) + \ldots$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of z = a.

4.10.2 Singularities of an Analytic Function

We have already defined a singular point of a function as the point at which the function ceases to be analytic.

1. **Isolated singularity.** If z = a is a singularity of f(z) such that f(z) is analytic at each point in its neighourhood (i.e. there exists a circle with centre a which has no other singularity), then z = a is called an **isolated singularity**.

   In such a case, f(z) can be expanded in a laurent's series around z = a, giving

   $$f(z) = a_0 + a_1(z-1) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots \quad \ldots (i)$$

   where, $$a_n = \frac{1}{2\pi i}\int \frac{f(t)}{(t-a)^{n+1}}\,dt$$

   For example. f(z) = cot (π/z) is not analytic where tan (π/z) = 0 i.e. at the points π/z = nπ or z = 1/n (n = 1, 2, 3....)

   Thus z = 1, 1/2, 1.3.... are all isolated singularities as there is no other singularity in their neighourhood

   But when n is large, z = 0 is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus z = 0 is the non-isolated singularity of f(z).

2. **Removable singularity.** If all the negative powers of (z – a) in (i) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$.

   Here the singularity can be removed by defining f(z) at z = a in such a way that it becomes analytic at z = a. Such a singularity is called a removable singularity.

   Thus if $\underset{z \to a}{\text{Lt}} f(z)$ exists finitely, then z = a is a removable singularity.

3. **Poles.** If all the negative powers of (z - a) in (i) after the nth are missing, then the singularity at z = a is called a pole of order n

4. **Essential singularity.** If the number of negative powers of (z – a) in (i) is infinite, then z = a is called an essential singularity. In this case, $\underset{z \to a}{\text{Lt}} f(z)$ does not exist.

ILLUSTRATIVE EXAMPLES

Example 1:

Poles and Essential singularities

The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

and

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}}$$

$$= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots$$

Note: The classification of singularities into poles and essential singularities is not merely a formal matter, because the behaviour of an analytic function in a neighborhood of an essential singularity is entirely from that in the neighborhood of a pole.

Example 2:

Find the nature of singularities of following functions

(a) $f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$ (b) $e^{\frac{1}{z}}$ (c) $\sin\frac{1}{z}$

Example 3:

Find the nature and location of singularities of the following functions:

(a) $\frac{z - \sin z}{z^2}$ (b) $(z+1)\sin\frac{1}{z-2}$ (c) $\frac{1}{\cos z - \sin z}$

Solution:

(a) Here $z = 0$ is a singularity.

Also
$$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right)\right\}$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \ldots$$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

(b)
$$(z+1)\sin\frac{1}{z-2} = (t+2+1)\sin\frac{1}{t}, \text{ where } t = z-2$$

$$= (t+3)\left\{\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \ldots\right\}$$

$$= \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \ldots\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5} - \ldots\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \ldots$$

$$= 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \ldots$$

Since there are infinite number of terms in the negative powers of $(z - 2)$, $z = 2$ is an essential singularity.

(c) Poles of $f(z) = \frac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, i.e. by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a simple pole of $f(z)$.

Example: 4

What type of singularity have the following functions:

(a) $\frac{1}{1-e^z}$ (b) $\frac{e^{2z}}{(z-1)^4}$ (c) ze^{1/z^2}

Solution:

(a) Poles of $f(z) = 1/(1 - e^z)$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore$ $z = 2n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$

Clearly $f(z)$ has a simple pole at $z = 2\pi i$.

(b)

$$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4} \cdot e^{2t}, \text{ where } t = z - 1$$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \ldots\right\}$$

$$= e^2\left\{\frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \ldots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \ldots\right\}$$

Since there are finite (4) number of terms containing negative powers of $(z - 1)$,

$\therefore$ $z = 1$ is a pole of 4th order.

(c)

$$f(z) = ze^{1/z^2}$$

$$= z\left\{1 + \frac{1}{1!\,z^2} + \frac{1}{2!\,z^4} + \frac{1}{3!\,z^6} + \ldots\right\}$$

$$= z + z^{-1} + \frac{z^{-3}}{2} + \frac{z^{-5}}{6} + \ldots \infty$$

Since there are infinite number of terms in the negative powers of z, therefore $z = 0$ is an essential singularity of $f(z)$.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.36 The analytic function $f(z) = \frac{z-1}{z^2+1}$ has singularities at

(a) 1 and – 1
(b) 1 and i
(c) 1 and – i
(d) i and – i

[CE, GATE-2009, 1 mark]

Solution: (d)

$$f(z) = \frac{z-1}{z^2+1} = \frac{z-1}{z^2-i^2} = \frac{z-1}{(z-i)(z+i)}$$

$\therefore$ The singularities are at $z = i$ and $-i$

Q.37 The value of the integral $\oint_c \frac{-3z+4}{(z^2+4z+5)} dz$ where c is the circle $|z| = 1$ is given by

(a) 0
(b) 1/10
(c) 4/5
(d) 1

[EC GATE-2011, 1 mark]

Solution: (a)

$$I = \oint_c \frac{-3z+4}{(z^2+4z+5)} dz = 2\pi i(\text{sum of residues})$$

Poles of $\frac{-3z+4}{(z^2+4z+5)}$ are given by

$$z^2 + 4z + 5 = 0$$

$$z = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

Since the poles lie outside the circle $|z| = 1$.
So f(z) is analytic inside the circle $|z| = 1$.

Hence $\oint_c f(z)dz = 2\pi i(0) = 0$

Q.38 The Taylor series expansion of 3sinx + 2cos x is _____.

(a) $2+3x-x^2-\frac{x^3}{2}+\cdots$
(b) $2-3x+x^2-\frac{x^3}{2}+\cdots$
(c) $2+3x+x^2+\frac{x^3}{2}+\cdots$
(d) $2-3x-x^2+\frac{x^3}{2}+\cdots$

[EC, GATE-2014 : 2 Marks, Set-1]

Solution : (a)

The Taylor's series expansion for

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \quad -\infty < x < \infty$$

and

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\cdots \quad -\infty < x < \infty$$

$$\therefore \quad 3\sin x + 2\cos x = 2+3x-\frac{2x^2}{2!}-\frac{3x^3}{3!}+\frac{2x^4}{4!}+\frac{3x^5}{5!}\cdots$$

$$= 2+3x-x^2-\frac{x^3}{2}+\cdots$$

Q.39 The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to

(a) $2 \ln 2$
(b) $\sqrt{2}$
(c) 2
(d) e

Solution : (d) [EC, GATE-2014 : 1 Mark, Set-4]

Given

Let
$$x(n) = \sum_{n=0}^{\infty} \frac{1}{n!}$$
$$= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

Also we know that expression of e^x
$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

Put
$$x = 1 \text{ in above expression}$$
$$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$
$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

4.11 RESIDUES

The coefficient of $(z - a)^{-1}$ in the expansion of f(z) around an isolated singularity is called the residue of f(z) at that point. Thus is the Laurent's series expansion of f(z) around z = a i.e. $f(z) = a_0 + a_1 (z - a) + a_2(z - a)^2 + \ldots + a_{-1}(z - a)^{-1} + a_{-2}(z - a)^{-2} + \ldots\ldots$, the residue of f(z) at z = a is a_{-1}.

Since,
$$a_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z - a)^{n+1}} dz$$

$$\therefore \quad a_{-1} = \text{Res } f(a) = \frac{1}{2\pi i} \int_C f(z) dz$$

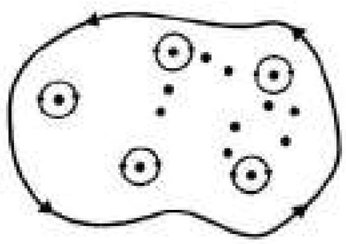
$$\therefore \quad \int_C f(z) dz = 2\pi i \text{ Res } f(a) \qquad \ldots \text{(i)}$$

4.11.1 Residue Theorem

If f(z) is analytic in a closed curve C except at a finite number of singular points within C, then

$$\int_C f(z) dz = 2\pi i \times (\text{sum of the residues at the singular points within C})$$

Let us surround each of the singular points $a_1, a_2, \ldots, a_n$ by a small circle such that it encloses no other singular point. Then these circles $C_1, C_2, \ldots, C_n$ together with C, form a multiply connected region in which f(z) is analytic.



∴ Applying Cauchy's theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots \int_{C_1} f(z)dz \quad \text{[by (i)]}$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots\ldots + \text{Res } f(a_n)]$$

which is the desired result.

4.11.2 Calculation of Residues

1. If f (z) has a simple pole at z = a, then

$$\textbf{Res } f(a) = \underset{z\to a}{\text{Lt}}\,[(z-a)f(z)] \quad \ldots (i)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 \ldots\ldots + c_{-1}(z-a)^{-1}$$

Multiplying throughout by z − a, we have

$$(z-a)\,f(z) = c_0(z-a) + c_1(z-a)^2 + \ldots + c_{-1}$$

Taking limits as z → a, we get

$$\underset{z\to a}{\text{Lt}}\,[(z-a)f(z)] = c_{-1} = \text{Res } f(a).$$

2. Another formula for Res f(a):

Let $f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z-a)\,F(z)$, $F(a) \neq 0$.

Then
$$\underset{z\to a}{\text{Lt}}\,[(z-a)\phi(z)/\psi(z)] = \underset{z\to a}{\text{Lt}}\,\frac{(z-a)[\phi(a)+(z-a)\phi'(a)+\ldots\ldots]}{\psi(a)+(z-a)\psi'(a)+\ldots\ldots}$$

$$= \underset{z\to a}{\text{Lt}}\,\frac{\phi(a)+(z-a)\phi'(a)+\ldots\ldots}{\psi'(a)+(z-a)\psi''(a)+\ldots\ldots}, \text{ since } \gamma(a) = 0$$

Thus,
$$\textbf{Res } f(a) = \frac{\phi(a)}{\psi'(a)}$$

3. If f(z) has a pole of order n at z = a, then

$$\textbf{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

Obs. In many cases, the residue of a pole (z = a) can be found , by putting z = a + t in f(z) and expanding it in powers of t where | t | is quite small.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.40 Consider likely applicability of Cauchy's Integral Theorem to evaluate the following integral counter clockwise around the unit circle c.

$$I = \oint_c \sec z\,dz,$$

z being a complex variable. The value of I will be

(a) I = 0 : singularities set = ϕ

(b) I = 0 : singularities set = $\left\{\pm\frac{2n+1}{2}\pi;\ n = 0, 1, 2\ldots\ldots\ldots\right\}$

(c) I = π/2 : singularities set = $\{\pm n\pi;\ n = 0, 1, 2\ldots\ldots\ldots\}$

(d) None of above

[CE, GATE-2005, 2 marks]

Solution: (a)

$$\int \sec z \, dz = \int \frac{1}{\cos z} dz$$

The poles are at
$$z_0 = (n+1/2)\pi = \ldots -3\pi/2, -\pi/2, \pi/2, +3\pi/2 \ldots\ldots$$

None of these poles lie inside the unit circle $|z| = 1$

Hence, sum of residues at poles = 0

$\therefore$ singularities set = ϕ and

$$I = 2\pi i \text{ [sum of residues of } f(z) \text{ at the poles]}$$
$$= 2\pi i \times 0 = 0$$

Q.41 Consider the following complex function

$$f(z) = \frac{9}{(z-1)(z+2)^2}$$

Which of the following is one of the residues of the above function?

(a) -1

(b) $\frac{9}{16}$

(c) 2

(d) 9

[CE, GATE-2015 : 2 Marks, Set-I]

Solution: (a)

$f(z)$ has poles at $z = 1, -2$

Residue of $f(z)$ at $(z = 1)$

$$= \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{9}{(z+2)^2} = 1$$

Residue of $f(z)$ at $(z = -2)$

$$= \lim_{z \to -2} \frac{d}{dz^{2-1}} \left[(z+2)^2 f(z) \right] = \lim_{z \to -2} \frac{d}{dz} \left(\frac{9}{z-1} \right)$$

$$= \lim_{z \to -2} \frac{-9}{(z-1)^2} = -1$$

Q.42 The integral $\oint f(z) dz$ evaluated around the unit circle on the complex plane for $f(z) = \frac{\cos z}{z}$ is

(a) $2\pi i$

(b) $4\pi i$

(c) $-2\pi i$

(d) 0

[ME, GATE-2008, 2 marks]

Solution: (a)

$$f(z) = \frac{\cos z}{z}$$

has simple pole at $z = 0$ and $z = 0$ is inside unit circle on complex plane

$\therefore$ Residue of $f(z)$ at $z = 0$

$$\underset{z \to 0}{\text{Lt}} f(z) \cdot z \cdot = \underset{z \to 0}{\text{Lt}} \cos z = 1$$

$$\int_C f(z) dz = 2\pi i \text{ (Residue at } z = 0) = 2\pi i . 1 = 2\pi i$$

Q.43 The value of the contour integral $\oint_{|z-i|=2} \frac{1}{z^2+4} dz$ in positive sense is

(a) $i\pi/2$ (b) $-\pi/2$

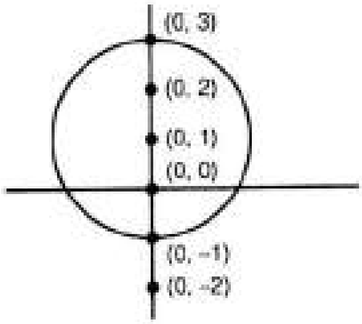
(c) $-i\pi/2$ (d) $\pi/2$ [EC, GATE-2006, 2 marks]

Solution: (d)

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

Pole (0, 2) lies inside the circle $|z-i| = 2$

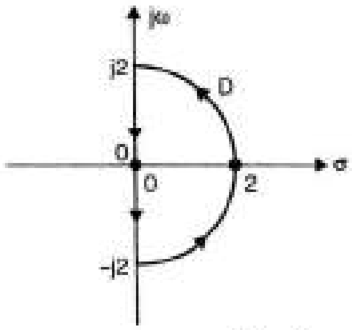
while pole (0, −2) is outside the circle $|z-i| = 2$ as can be seen from figure below:



$$\int_C f(z)\,dz = 2\pi i\,[\text{Residue at those poles which are inside C}]$$

$$= 2\pi i \text{ Res } f(2i) = 2\pi i \frac{f}{(2i+2i)} = \frac{\pi}{2}$$

Q.44 If the semi-circular contour D of radius 2 is as shown in the figure, then the value of the integral $\oint_D \frac{1}{(s^2-1)} ds$ is



(a) $j\pi$ (b) $-j\pi$

(c) $-\pi$ (d) π [EC, GATE-2007, 2 marks]

Solution: (a)

$$I = \oint \frac{1}{(s^2-1)} ds = \oint \frac{1}{(s+1)(s-1)} ds = 2\pi j \times (\text{Sum of residues})$$

pole $s = -1$ is not inside the contour D, but $s = 1$ is inside D

residue at pole $s = 1$ is

$$z = \lim_{s \to 1} \frac{(s-1)}{(s-1)(s+1)} = \frac{1}{2} \quad \Rightarrow \oint \frac{1}{(s^2-1)} ds = 2\pi j \times \frac{1}{2} = j\pi$$

Q.45 The residue of the function $f(z) = \frac{1}{(z+2)^2(z-2)^2}$ at $z = 2$ is

(a) $-\frac{1}{32}$ (b) $-\frac{1}{16}$

(c) $\frac{1}{16}$ (d) $\frac{1}{32}$ [EC, GATE-2008, 2 marks]

Solution: (a)

Since $\underset{z \to 2}{Lt}\left[(z-2)^2 f(z)\right]$ is finite and non-zero, f(z) has a pole of order two at z = 2.

The residue at z = a is given for a pole of order n as

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=2}$$

Here n = 2 (pole of order 2) and a = 2

$$\therefore \quad \text{Res } f(2) = \frac{1}{1!}\left\{\frac{d}{dz}\left[(z-2)^2 f(z)\right]\right\}_{z=2}$$

$$= \left\{\frac{d}{dz}\left[(z-2)^2 \frac{1}{(z+2)^2 (z-2)^2}\right]\right\}_{z=2}$$

$$= \left\{\frac{d}{dz}\left[\frac{1}{(z+2)^2}\right]\right\}_{z=2} = \left[-2(z+2)^{-3}\right]_{z=2} = \frac{-2}{(2+2)^3} = -\frac{1}{32}$$

Q.46 If $f(z) = c_0 + c_1 z^{-1}$, then $\oint_{\text{unit circle}} \frac{1+f(z)}{z} dz$ is given by

(a) $2\pi c_1$ (b) $2\pi(1+c_0)$

(c) $2\pi j c_1$ (d) $2\pi j(1+c_0)$ [EC, GATE-2009, 1 mark]

Solution: (d)

$$f(z) = c_0 + c_1 z^{-1}$$

$$\oint \frac{1+f(z)}{z} dz = ?$$

It has one pole at origin, which is inside unit circle

So, $$\oint \frac{[1+f(z)]}{z} dz = 2\pi j \quad \text{[Residue of f(z) at z = 0]}$$

$$= 2\pi j\,[1 + f(0)]$$

Since, $$f(z) = C_0 + C_1 z^{-1} \Rightarrow f(0) = C_0$$

$$\therefore \quad \text{Answer} = 2\pi j\,(1 + C_0)$$

Q.47 The residues of a complex function $X(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles are

(a) $\frac{1}{2}, -\frac{1}{2}$ and 1 (b) $\frac{1}{2}, \frac{1}{2}$ and -1

(c) $\frac{1}{2}$, 1 and $-\frac{3}{2}$ (d) $\frac{1}{2}$, -1 and $\frac{3}{2}$

[EC, GATE-2010, 2 marks]

Solution: (c)

$$x(z) = \frac{1-2z}{z(z-1)(z-2)}$$

poles are z = 0, z = 1 and z = 2

Residue at z = 0

$$\text{residue} = \text{value of } \frac{1-2z}{(z-1)(z-2)} \text{ at } z = 0$$

$$= \frac{1-2\times 0}{(0-1)(0-2)} = \frac{1}{2}$$

Residue at z = 1

$$\text{residue} = \text{value of } \frac{1-2z}{z(z-2)} \text{ at } z = 1 = \frac{1-2\times 1}{1(1-2)} = 1$$

Residue at z = 2

$$\text{residue} = \text{value of } \frac{1-2z}{z(z-1)} \text{ at } z = 2$$

$$= \frac{1-2\times 2}{2(2-1)} = -\frac{3}{2}$$

∴ The residues at its poles are $\frac{1}{2}$, 1 and $-\frac{3}{2}$.

Q.48 The value of the integral $\frac{1}{2\pi j}\int_C \frac{z^2+1}{z^2-1}dz$ where z is a complex number and C is a unit circle with center at $1 + 0j$ in the complex plane is______.

[IN, 2016 : 2 Marks]

Solution:

$$\frac{1}{2\pi j}\int_C \frac{z^2+1}{(z^2-1)}dz = \frac{1}{2\pi j}\int_C \frac{z^2+1}{(z-1)(z+1)}dz$$

Poles are at $z = 1, -1$

Given circle is $|z-1| = 1$

pole $z = 1$ lies inside C

pole $z = -1$ lies outside C

Res f(z) at $z = 1$ is

$$= \underset{z\to 1}{Lt}(z-1)\frac{z^2+1}{(z-1)(z+1)} = \frac{2}{2} = 1$$

Res $f(z)$ at $z = -1$ is $= 0$

By Cauchy's residue theorem

$$\frac{1}{2\pi j}\int_C \frac{z^2+1}{z^2-1}dz = \frac{1}{2\pi j}\times 2\pi j(1+0) = 1$$

Q.49 In the following integral, the contour C encloses the points $2\pi j$ and $-2\pi j$

$-\frac{1}{2\pi}\oint_C \frac{\sin z}{(z-2\pi j)^3}dz$. The value of the integral is _____

[EC, 2016 : 2 Marks, Set-1]

Solution:

$$I = -\frac{1}{2\pi}\int_c \frac{\sin z}{(z-2\pi j)^3}dz = -\frac{1}{2\pi}\times\frac{2\pi j f''(2\pi j)}{2!}$$

$$f(z) = \sin z$$
$$f'(z) = \cos z$$
$$f''(z) = -\sin z$$

$$I = -\frac{1}{2\pi}\times 2\pi j\frac{-\sin(2\pi j)}{2} = -\frac{1}{2}\sinh 2\pi = -133.87$$

Q.50 The values of the integral $\frac{1}{2\pi j}\oint_c \frac{e^z}{z-2}dz$ along a closed contour c in anti-clockwise direction for

(i) the point $z_0 = 2$ inside the contour c, and

(ii) the point $z_0 = 2$ outside the contour c, respectively, are

(a) (i) 2.72, (ii) 0
(b) (i) 7.39, (ii) 0
(c) (i) 0, (ii) 2.72
(d) (i) 0, (ii) 7.39

[EC, 2016 : 2 Marks, Set-3]

Solution: (b)

(i) $Z_0 = 2$ lies inside C,

so $\text{Res } f(z) = \lim_{z\to 2}(z-2)\cdot\frac{e^z}{z-2}$

$= e^2 = 7.39$

$$\frac{1}{2\pi i}\int_c \frac{e^z}{z-2}dz = 2\pi i\cdot\frac{1}{2\pi i}(7.39) = 7.39$$

(ii) $Z_0 = -2$ lies out side C then

$\text{Res } f(z) = 0$

so $\int_c \frac{e^z}{z-2}dz = 2\pi i\frac{1}{2\pi i}(0) = 0\{\ \}$

Q.51 For $f(z) = \frac{\sin(z)}{z^2}$, the residue of the pole at $z = 0$ is ________.

[EC, 2016 : 1 Mark, Set-3]

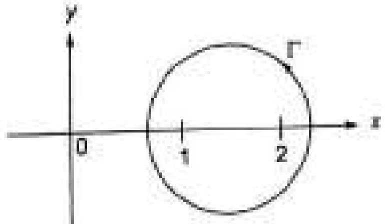
Solution:

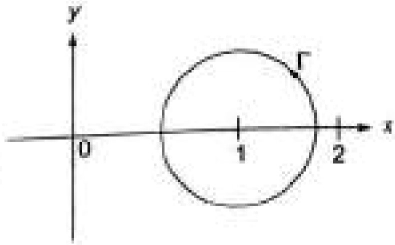
Residue of $\frac{\sin z}{z^2}$ $= \text{coefficient of } \frac{1}{z} \text{ in } \left\{\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - - -}{z^2}\right\}$

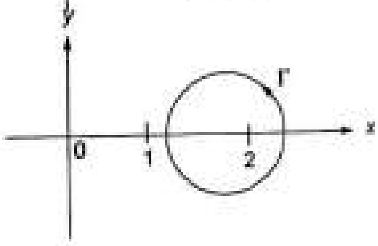
$= \text{coefficient of } \frac{1}{z} \text{ in } \left\{\frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - - -\right\}$

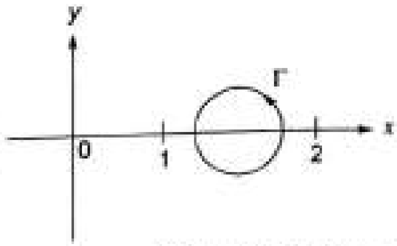
$= 1$

Q.52 The value of $\oint_\Gamma \frac{3z-5}{(z-1)(z-2)}dz$ along a closed path Γ is is equal to $(4\pi i)$, where $z = x + iy$ and $i = \sqrt{-1}$. The correct path Γ is

(a)


(b)


(c)


(d)


[ME, 2016 : 2 Marks, Set-2]

Solution: (b)

$$\int_\Gamma \frac{3z-5}{(z-1)(z-2)}dz = 4\pi i$$

$$\int_\Gamma \frac{3z-5}{(z-1)(z-2)}dz = 2\pi i(2)$$

Sum of residues must be equal to 2.

$$\underset{z=1}{\text{Res}\, f(z)} = \underset{z\to 1}{u}(z-1)\frac{3z-5}{(z-1)(z-2)} = \frac{-2}{-1} = 2$$

$$\underset{z=2}{\text{Res}\, f(z)} = \underset{z\to 2}{u}(z-2)\frac{3z-5}{(z-1)(z-2)} = \frac{6-5}{2-1} = 1$$

Therefore $z = 1$ must lies inside C

$z = 2$ lies outside C

then only we will get the given integral values is equal to $L\pi i$.

OOOO

CHAPTER 5

Probability and Statistics

5.1 PROBABILITY FUNDAMENTALS

5.1.1 Definitions

Sample Space and Event: Consider an experiment whose outcome is not predictable with certainty. Such an experiment is called a **random experiment**. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the **sample space** of experiment and is denoted by S. Some examples follow.

1. If the outcome of an experiment consist in the determination of the sex of a newborn child, then S = {g, b} where the outcome g means that the child is a girl and b is the boy.
2. If the outcome of an experiment consist of what comes up on a single dice, then S = {1, 2, 3, 4, 5, 6}.
3. If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, 7; then S = {all 7! permutations of the (1, 2, 3, 4, 5, 6, 7)}.

The outcome (2, 3, 1, 6, 5, 4, 7) means, for instances, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

Any subset E of the sample space is known as **Event**. That is, an event is a set consisting of some or all of the possible outcomes of the experiment. For example, in the throw of a single dice S = {1, 2, 3, 4, 5, 6} and some possible events are

$$E_1 = \{1, 2, 3\}$$
$$E_2 = \{3, 4\}$$
$$E_3 = \{1, 4, 6\} \text{ etc.}$$

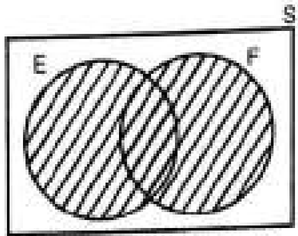
If the outcome of the experiment is contained in E, then we say that E has occurred. Always $E \subseteq S$. Since E & S are sets, theorems of set theory may be effectively used to represent & solve probability problems which are more complicated.

Example: If by throwing a dice, the outcome is 3, then events E_1 and E_2 are said to hare occured.

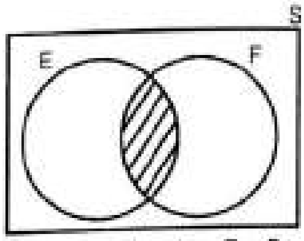
In the child example – (i) If E_1 = {g}, then E_1 is the event that the child is a girl.

Similarly, if E_2 = {b}, then E_2 is the event that the child is a boy. These are examples of **Simple** events. **Compound** events may consist of more than one outcome. Such as E = {1, 3, 5} for an experiment of throwing a dice. We say event E has happened if the dice comes up 1 or 3 or 5.

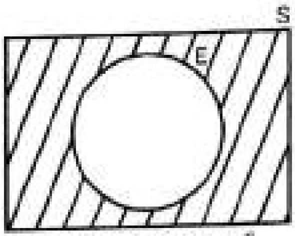
For any two events E and F of a sample space S, we define the new event $E \cup F$ to consists of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if either E or F or both occurs. For instances, in the dice example (i) if event E = {1, 2} and F = {3, 4}, then $E \cup F$ = {1, 2, 3, 4}.
That is $E \cup F$ would be another event consisting of 1 or 2 or 3 or 4. The event $E \cup F$ is called **union** of event E and the event F. Similarly, for any two events E and F we may also define the new event $E \cap F$, called **intersection** of E and F, to consists of all outcomes that are common to both E and F.



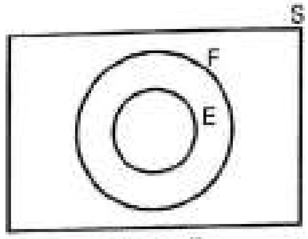
(a) Shaded region : $E \cup F$



(b) Shaded region : $E \cap F$



(c) Shaded region : E^c



(d) $E \subset F$

5.1.2 Types of Events

5.1.2.1 Complementary Event

The event E^C is called complementary event for the event E. It consists of all outcomes not in E, but in S. For example, in a dice throw, if E = {Even nos} = {2, 4, 6} then E^C = {Odd nos} = {1, 3, 5}.

5.1.2.2 Equally Likely Events

Two events E and F are equally likely iff

$$p(E) = p(F)$$

For example,

$$E = \{1, 2, 3\}$$
$$F = \{4, 5, 6\}$$

are equally likely, since $p(E) = p(F) = 1/2$.

5.1.2.3 Mutually Exclusive Events

Two events E and F are mutually exclusive, if $E \cap F = \phi$ i.e. $p(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).

5.1.2.4 Collectively Exhaustive Events

Two events E and F are collectively exhaustive, if $E \cup F = S$ i.e. together E and F include all possible outcomes, $p(E \cup F) = p(S) = 1$.

5.1.2.5 Independent Events

Two events E and F are independent iff

$$p(E \cap F) = p(E) \cdot p(F)$$

Also $p(E \mid F) = p(E)$ and $p(F \mid E) = p(F)$.

Whenever E and F are independent. i.e. when two events E and F are independent, the conditional probability becomes same as marginal probability. i.e. probability E is not affected by whether F has happened or not, and viceversa i.e., when E is independent of F, then F is also independent of E.

5.1.3 DeMorgan's Law

1. $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$

2. $\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$

Example:

$$(E_1 \cup E_2)^C = E_1^C \cap E_2^C$$
$$(E_1 \cap E_2)^C = E_1^C \cup E_2^C$$

Note that $E_1^C \cap E_2^C$ is the event **neither E_1 nor E_2**.

$E_1 \cup E_2$ is the event **either E_1 or E_2 (or both).**

Demorgan's law is often used to find the probability of neither E_1 nor E_2.

i.e. $p(E_1^C \cap E_2^C) = p[(E_1 \cup E_2)^C] = 1 - p(E_1 \cup E_2)$.

5.1.4 Approaches to Probability

There are 2 approaches to quantifying probability of an Event E.

1. **Classical Approach:**

$$P(E) = \frac{n(E)}{n(S)} = \frac{|E|}{|S|}$$

i.e. the ratio of number of ways an event can happen to the number of ways sample space can happen, is the probability of the event. Classical approach assumes that all outcomes are equally likely.

ILLUSTRATIVE EXAMPLES

Example:

If out all possible jumbles of the word "BIRD", a random word is picked, what is the probability, that this word will start with a "B".

Solution:

$$p(E) = \frac{n(E)}{n(S)}$$

In this problem

$n(S)$ = all possible jumbles of BIRD = 4!

$n(E)$ = those jumbles starting with "B" = 3!

So,

$$p(E) = \frac{n(E)}{n(S)} = \frac{3!}{4!} = \frac{1}{4}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 Suppose we uniformly and randomly select a permutation from the 20! permutations of 1, 2, 3 ...., 20. What is the probability that 2 appears at an earlier position that any other even number in the selected permutation?

(a) $\frac{1}{2}$ (b) $\frac{1}{10}$

(c) $\frac{9!}{20!}$ (d) None of these

[CS, GATE-2007, 2 marks]

Solution: (d)

Number of permutations with '2' in the first position = 19!

Number of permutations with '2' in the second position = 10 × 18!

(fill the first space with any of the 10 odd numbers and the 18 spaces after the 2 with 18 of the remaining numbers in 18! ways)

Number of permutations with '2' in 3rd position = 10 × 9 × 17!

(fill the first 2 places with 2 of the 10 odd numbers and then the remaining 17 places with remaining 17 numbers)

and so on until '2' is in 11th place. After that it is not possible to satisfy the given condition, since there are only 10 odd numbers available to fill before the '2'. So the desired number of permutations which satisfies the given condition is

$19! + 10 \times 18! + 10 \times 9 \times 17! + 10 \times 9 \times 8 \times 16! + \ldots + 10! \times 9!$

Now the probability of this happening is given by

$$\frac{19! + 10 \times 18! + 10 \times 9 \times 17! \ldots + 10! \times 9!}{20!}$$

Which is clearly not choices (a), (b) or (c). ∴ Answer is (d)-none of these.

Q.2 A deck of 5 cards (each carrying a distinct number from 1 to 5) is shuffled thoroughly. Two cards are then removed one at a time from the deck. What is the probability that the two cards are selected with the number on the first card being one higher than the number on the second card?

(a) 1/5 (b) 4/25

(c) 1.4 (d) 2/5 **[CS, GATE-2011, 2 marks]**

Solution: (a)

The five cards are {1, 2, 3, 4, 5}

Sample space = 5 × 4 ordered pairs.

[Since there is a Ist card and IInd card we have to take ordered pairs]

p(Ist card = IInd card + 1)

$$= P\{(2, 1), (3, 2), (4, 3), (5, 4)\} = \frac{4}{5 \times 4} = \frac{1}{5}$$

2. **Frequency Approach:** Since sometimes all outcomes may not be equally likely, a more general approaches is the frequency approach, where probability is defined as the relative frequency of occurrence of E.

$$P(E) = \lim_{N \to \infty} \frac{n(E)}{N}$$

where N is the number of times exp is performed and n(E) is the no of times the event E occurs.

ILLUSTRATIVE EXAMPLES

Example:

From the following table find the probability of obtaining "A" grade in this exam.

Grade	A	B	C	D
No. of Students	10	20	30	40

Solution:

By frequency approach,

$$N = \text{total no of students} = 100$$

$$p(\text{A grade}) = \frac{n(\text{A grade})}{N} = \frac{10}{100} = 0.1$$

5.1.5 Axioms of Probability

Consider an experiment whose sample space is S. For each event E of the sample space S we assume that a number P(E) is defined and satisfies the following three axioms.

Axiom-1: $0 \le P(E) \le 1$

Axiom-2: $P(S) = 1$

Axiom-3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_i \cap E_j = \phi$ when $i \ne j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Example: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ where (E_1, E_2 are mutually exclusive).

5.1.6 Rules of Probability

There are six rules of probability using which probability of any compound event involving arbitrary events A and B, can be computed.

Rule 1:

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

This rule is also called the inclusion-exclusion principle of probability.

This formula reduces to

$$p(A \cup B) = p(A) + p(B)$$

if A and B are mutually exclusive, since $p(A \cap B) = 0$ in such a case.

Rule 2:

$$p(A \cap B) = p(A) \cdot p(B/A) = p(B) \cdot p(A/B)$$

where p(A/B) represents the **conditional probability of A given B** and p(B/A) represents the conditional probability of B given A.

(a) p(A) and p(B) are called the **marginal probabilities** of A and B respectively. This rule is also called as the multiplication rule of probability.

(b) $p(A \cap B)$ is called the **joint probability** of A and B.

(c) If A and B are **independent** events, this formula reduces to

$$p(A \cap B) = p(A) \cdot p(B).$$

since when A and B are independent

$$p(A/B) = p(A)$$

and

$$p(B/A) = p(B)$$

i.e. the conditional probabilities become same as the marginal (unconditional) probabilities.

(d) If A and B are independent, then so are A and B^C; A^C and B and A^C and B^C.

(e) Condition for three events to independent:

Events A, B and C are independent iff

$$p(ABC) = p(A)\,p(B)\,p(C)$$

and $p(AB) = p(A)\,p(B)$

and $p(AC) = p(A)\,p(C)$ — A, B, C are pairwise independent

and $p(BC) = p(B)\,p(C)$

Note: If A, B, C are independent, then A will be independent of any event formed from B and C. For instance, A is independent of $B \cup C$.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.3 There are two containers, with one containing 4 red and 3 green balls and the other containing 3 blue and 4 green balls. One ball is drawn at random from each container. The probability that one of the balls is red and the other is blue will be

(a) 1/7 (b) 9/49
(c) 12/49 (d) 3/7

[CE, GATE-2011, 1 mark]

Solution: (c)

$$p(\text{one ball is Red \& another is blue}) = p(\text{first is Red and second is Blue})$$
$$= \frac{4}{7} \times \frac{3}{7} = \frac{12}{49}$$

Q.4 If P and Q are two random events, then the following is TRUE

(a) Independence of P and Q implies that probability $(P \cap Q) = 0$

(b) Probability $(P \cup Q) \geq$ Probability (P) + Probability (Q)

(c) If P and Q are mutually exclusive, then they must be independent

(d) Probability $(P \cap Q) \leq$ Probability (P)

[EE, GATE-2005, 1 mark]

Solution: (d)

(a) is false since of P and Q are independent

$$pr(P \cap Q) = pr(P) * pr(Q)$$

which need not be zero.

(b) is false since $pr(P \cup Q) = pr(P) + pr(Q) - pr(P \cap Q)$

$\therefore \quad pr(P \cup Q) \leq pr(P) + pr(Q)$

(c) is false since independence and mutually exclusion are unrelated properties.

(d) is true

since $P \cap Q \subseteq P$

$\Rightarrow \quad n(P \cap Q) \leq n(P)$

$\Rightarrow \quad pr(P \cap Q) \leq pr(P)$

Q.5 A loaded dice has following probability distribution of occurrences

DiceValue	1	2	3	4	5	6
Probability	1/4	1/8	1/8	1/8	1/8	1/4

If three identical dice as the above are thrown, the probability of occurrence of values 1, 5 and 6 on the three dice is

(a) same as that of occurrence of 3, 4, 5 (b) same as that of occurrence of 1, 2, 5
(c) 1/128 (d) 5/8

[EE, GATE-2007, 2 marks]

Solution: (c)

Dice value	1	2	3	4	5	6
Probability	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

Since the dice are independent,

$$p(1, 5, 6) = \frac{1}{4}\times\frac{1}{8}\times\frac{1}{4} = \frac{1}{128}$$

Q.6 A fair dice is rolled twice. The probability that an odd number will follow an even number is

(a) $\frac{1}{2}$
(b) $\frac{1}{6}$
(c) $\frac{1}{3}$
(d) $\frac{1}{4}$

[EC, GATE-2005, 1 mark]

Solution: (d)

$$P_o = \frac{3}{6} = \frac{1}{2}$$

$$P_e = \frac{3}{6} = \frac{1}{2}$$

Since both events are independent of each other.

$$P_{(odd/even)} = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

Q.7 An examination consists of two papers, Paper 1 and Paper 2. The probability of failing in Paper 1 is 0.3 and that in Paper 2 is 0.2. Given that a student has failed in Paper 2, the probability of failing in Paper 1 is 0.6. The probability of a student failing in both the papers is

(a) 0.5
(b) 0.18
(c) 0.12
(d) 0.06

[EC, GATE-2007, 2 marks]

Solution: (c)

(A denote the event of failing in paper 1)
(B denote the event of failing in paper 2)

Given, $P(A) = 0.3$
$P(B) = 0.2$
$P(A/B) = 0.6$

Probability of failing in both

$$P(A \cap B) = P(B) * p(A \mid B)$$
$$= 0.2 * 0.6 = 0.12$$

Q.8 A fair coin is tossed 10 times. What is the probability that ONLY the first two tosses will yield heads?

(a) $\left(\frac{1}{2}\right)^2$
(b) ${}^{10}C_2\left(\frac{1}{2}\right)^2$
(c) $\left(\frac{1}{2}\right)^{10}$
(d) ${}^{10}C_2\left(\frac{1}{2}\right)^{10}$

[EC, GATE-2009, 1 mark]

Solution: (c)

p(only first two tosses are heads) = p(H, H, T, T, T, ... T)

Now, each toss is independent.

So required probability

$$= p(H) \times p(H) \times [p(T)]^8 \ldots$$

$$= \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^{10}$$

Q.9 A fair dice is tossed two times. The probability that the second toss results in a value that is higher than the first toss is

(a) 2/36
(b) 2/6
(c) 5/12
(d) 1/2

[EC, GATE-2011, 1 mark]

Solution: (c)

$$\text{Total cases} = 36 \quad [(1, 1)(1, 2)(1, 3) \text{ and so on}]$$

$$\text{Favourable case} = (x_1 > x_2) = 15$$

$$P[x_1 > x_2] = \frac{15}{36} = \frac{5}{12}$$

Q.10 Suppose A and B are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\bar{A}$ and $\bar{B}$ be their complements. Which one of the following statements is FALSE?

(a) $P(A \cap B) = P(A)\,P(B)$
(b) $P(A/B) = P(A)$
(c) $P(A \cup B) = P(A) + P(B)$
(d) $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$

[EC, GATE-2015 : 1 Mark, Set-1]

Solution: (c)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since $P(A \cap B) = p(A)\,p(B)$ (not necessarily equal to zero).

So, $P(A \cup B) = P(A) + P(B)$ is false.

Q.11 A fair coin is tossed till a head appears for the first time. The probability that the number of required tosses is odd, is

(a) 1/3
(b) 1/2
(c) 2/3
(d) 3/4

[EC, EE, IN, GATE-2012, 2 marks]

Solution: (c)

$$P(H) = \frac{1}{2} \quad ; \quad P(T) = \frac{1}{2}$$

Favourable situation: H or TTH or TTTT H and so on

$$\text{Probability of odd number of tosses} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \ldots$$

$$= \frac{1}{2}\left[1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-1/4}\right] = \frac{2}{3} \quad \text{(sum of infinite geometric series with a = 1 and r = 1/4)}$$

Q.12 Suppose a fair six-sided die ti rolled once. If the value on the die is 1, 2 or 3 the die is rolled a second time. What is the probability that the sum of total values that turn up is at least 6?

(a) 10/21
(b) 5/12
(c) 2/3
(d) 1/6

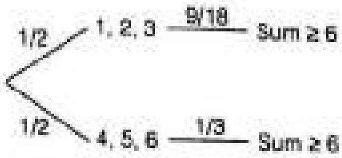
[CS, GATE-2012, 2 marks]

Solution: (b)

If first throw is 1, 2 or 3 then sample space is only 18 possible ordered pairs. Out of this only (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5) and (3, 6) i.e. 9 out of 18 ordered pairs gives a Sum ≥ 6.

If first throw is 4, 5 or 6 then second throw is not made and therefore the only way Sum ≥ 6 is if the throw was 6. Which is one out of 3 possible.

So the tree diagram becomes as follows:



From above diagram

$$P(\text{sum} \geq 6) = \frac{1}{2} \times \frac{9}{18} + \frac{1}{2} \times \frac{1}{3} = \frac{15}{36} = \frac{5}{12}$$

Q.13 Suppose X_i for $i = 1, 2, 3$ are independent and identically distributed random variables whose probability mass functions are $Pr[X_i = 0] = Pr[X_i = 1] = 1/2$ for $i = 1, 2, 3$. Define another random variable $Y = X_1X_2 \oplus X_3$, where ⊕ denotes XOR. Then $Pr[Y = 0 \mid X_3 = 0] =$ ______.

[CS, 2015 : 2 Marks, Set-3]

Solution: (0.75)

X_1	X_2	X_3	X_1X_2	$Y = X_1X_2 \oplus X_3$
0	0	0	0	0
0	1	0	0	0
1	0	0	0	0
1	1	0	1	1

Y = 0 and $X_3 = 0$ in 3 cases

$X_3 = 0$ in 4 cases

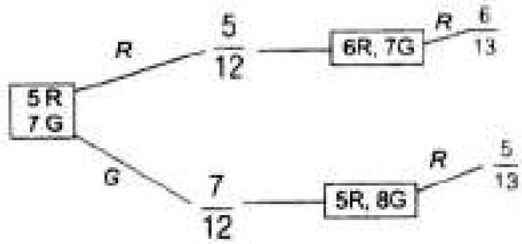
$$P(Y = 0 / X_3 = 0) = \frac{P(Y = 0 \cap X_3 = 0)}{P(X_3 = 0)} = \frac{3}{4} = 0.75$$

Q.14 An urn contains 5 red and 7 green balls. A ball is drawn at random and its colour is noted. The ball is placed back into the urn along with another ball of the same colour. The probability of getting a red ball in the next draw is

(a) $\frac{65}{156}$
(b) $\frac{67}{156}$
(c) $\frac{79}{156}$
(d) $\frac{89}{156}$

[IN, 2016 : 2 Marks]

Solution: (a)



$$P(\text{Red}) = \frac{5}{12}\times\frac{6}{13}+\frac{7}{12}\times\frac{5}{13}$$

$$= \frac{65}{156}$$

Rule 3: Complementary Probability

$$p(A) = 1-p(A^C)$$

$p(A^C)$ is called the complementary probability of A and $p(A^C)$ represents the probability that the event A will not happen.

$\therefore$ $$p(A) = 1-p(A^C)$$

$p(A^C)$ is also written as $p(A')$

Notice that $p(A) + p(A') = 1$

i.e. A and A′ are mutually exclusion as well as collectively exhaustive.

Also notice that by Demorgan's law since $A^C \cap B^C = (A\cup B)^C$

$$p(A^C\cap B^C) = p(A\cup B)^C = 1-p(A\cup B)$$

i.e. $$p(\text{neither A nor B}) = 1-p(\text{either A or B})$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.15 A single die is thrown twice. What is the probability that the sum is neither 8 nor 9?

(a) 1/9 (b) 5/36

(c) 1/4 (d) 3/4

[ME, GATE-2005, 2 marks]

Solution: (d)

$$\text{Sample space} = (6)^2 = 36$$

Total ways in which sum is either 8 or 9 is

(2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (5, 3), (5, 4), (6, 2), (6, 3) = 9 ways

$\therefore$ Probability of coming sum 8 or 9 = $\frac{9}{36}=\frac{1}{4}$

So probability of not coming sum 8 or 9 = $1-\frac{1}{4}=\frac{3}{4}$

Q.16 Three vendors were asked to supply a very high precision component. The respective probabilities of their meeting the strict design specifications are 0.8, 0.7 and 0.5. Each vendor supplies one component. The probability that out of total three components supplied by the vendors, at least one will meet the design specification is ______.

[ME, GATE-2015 : 1 Mark, Set-2]

Solution: (0.97)

Probability of atleast one meet the specification

$$= 1-\left(\bar{A}\times\bar{B}\times\bar{C}\right) = 1-(0.2\times0.3\times0.5) = 0.97$$

Q.17 The probability of getting a "head" in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a "head" is obtained. If the tosses are independent, then the probability of getting "head" for the first time in the fifth toss is ______.

[EC, 2016 : 1 Mark, Set-3]

Solution:

$$P(H) = 0.3$$
$$P(T) = 0.7$$

since all tosses are independent

so, probability of getting head for the first time in 5^{th} toss is

$$= P(T)\,P(T)\,P(T)\,P(T)\,P(H) = 0.7\times0.7\times0.7\times0.7\times0.3$$
$$= 0.072$$

Rule 4: Conditional Probability Rule

Starting from the multiplication rule

$$p(A\cap B) = p(B)\cdot p(A/B)$$

by cross multiplying we get the conditional probability formula

$$p(A/B) = \frac{p(A\cap B)}{p(B)}$$

By interchanging A and B in this formula we get

$$p(B/A) = \frac{p(A\cap B)}{p(A)}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.18 A hydraulic structure has four gates which operate independently. The probability of failure of each gate is 0.2. Given that gate 1 has failed, the probability that both gates 2 and 3 will fail is

(a) 0.240 (b) 0.200

(c) 0.040 (d) 0.008 **[CE, GATE-2004, 2 marks]**

Solution: (c)

Since all three gates are independent

p(gate 2 and gate 3 fail | gate 1 failed)

= p(gate 2 and gate 3 fail)

= p(gate 2) × p(gate 3) [gate 2 and 3 fail independently]

$= 0.2\times0.2 = 0.04$

Q.19 A person on a trip has a choice between private car and public transport. The probability of using a private car is 0.45. While using the public transport, further choices available are bus and metro, out of which the probability of commuting by a bus is 0.55. In such a situation, the probability (rounded up to two decimals) of using a car, bus and metro, respectively would be

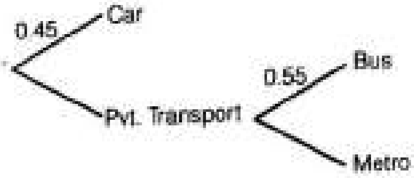
(a) 0.45, 0.30 and 0.25 (b) 0.45, 0.25 and 0.30

(c) 0.45, 0.55 and 0.00 (d) 0.45, 0.35 and 0.20

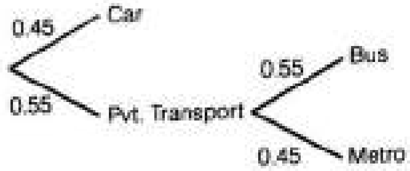
[CE, GATE-2008, 2 marks]

Solution: (a)

The information given in the problem can be represented by the tree diagram given below:



Now completing the blanks in the above diagram we have the final diagram as shown below:



From above diagram

$\therefore$ $p(\text{Car}) = 0.45$

$p(\text{Bus}) = 0.55 \times 0.55 = 0.30$

and $p(\text{Metro}) = 0.55 \times 0.45 = 0.25$

Q.20 A box contains 2 washers, 3 nuts and 4 bolts. Items are drawn from the box at random one at a time without replacement. The probability of drawing 2 washers first followed by 3 nuts and subsequently the 4 bolts is

(a) 2/315
(b) 1/630
(c) 1/1260
(d) 1/2520

[ME, GATE-2010, 2 marks]

Solution: (c)

Box contains 2 washers, 3 nuts and 4 bolts

p(2 washers, then 3 nuts, then 4 bolts)

$$= \left(\frac{2}{9}\times\frac{1}{8}\right)\times\left(\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\right)\times\left(\frac{4}{4}\times\frac{3}{3}\times\frac{2}{2}\times\frac{1}{1}\right) = \frac{1}{1260}$$

Q.21 The chance of a student passing an exam is 20%. The chance of a student passing the exam and getting above 90% marks in it is 5%. Given that a student passes the examination, the probability that the student gets above 90% marks is

(a) $\frac{1}{18}$
(b) $\frac{1}{4}$
(c) $\frac{2}{9}$
(d) $\frac{5}{18}$

[ME, GATE-2015 : 2 Marks, Set-2]

Solution: (b)

Given, $p(\text{passing the exam}) = 0.2$

$p(\text{passing the exam} \cap > 90\%) = 0.05$

The desired probability

$$= p(> 90\% \mid \text{passing the exam})$$

$$= \frac{p(\text{passing the exam} \cap > 90\%)}{p(\text{passing the exam})} = \frac{0.05}{0.2} = \frac{1}{4}$$

Q.22 If $P(X) = \frac{1}{4}$, $P(Y) = \frac{1}{3}$, and $P(X \cap Y) = \frac{1}{12}$, the value of $P(Y/X)$ is

(a) $\frac{1}{4}$ (b) $\frac{4}{25}$

(c) $\frac{1}{3}$ (d) $\frac{29}{50}$ **[ME, GATE-2015 : 1 Mark, Set-3]**

Solution: (c)

$$P\left(\frac{Y}{X}\right) = \frac{P(Y \cap X)}{P(X)} = \frac{1/12}{1/4} = \frac{1}{3}$$

Q.23 A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is

(a) 1/8 (b) 1/2

(c) 3/8 (d) 3/4 **[EE, GATE-2005, 2 marks]**

Solution: (b)

Sample space = {HHH, HTH, HHT, HTT}

Favourable (2 heads in 3 tosses) = {HTH, HHT}

$$\text{Required probability} = \frac{2}{4} = \frac{1}{2}$$

Q.24 Two fair dice are rolled and the sum r of the numbers turned up is considered

(a) Pr (r > 6) = (1/6) (b) Pr (r/3 is an integer) = (5/6)

(c) Pr (r = 8 | r/4 is an integer) = (5/9) (d) Pr (r = 6 | r/5 is an integer) = (1/18)

[EE, GATE-2006, 2 marks]

Solution: (c)

If two fair dices are rolles the probability distribution of r where r is the sum of the numbers on each die is given by

Y	2	3	4	5	6	7	8	9	10	11	12
P(y)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The above table has been obtained by taking all different ways of obtaining a particular sum. For example, a sum of 5 can be obtained by (1, 4), (2, 3), (3, 2) and (4, 1).

$\therefore \quad p(x = 5) = 4/36$

Now let us consider choice (a)

$$Pr(r > 6) = pr(r \geq 7)$$

$$= \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{21}{36} = \frac{7}{12}$$

$\therefore$ choice (a) pr (r > 6) = 1/6 is wrong.

Consider choice (b)

$$Pr\,(r/3 \text{ is an integer}) = pr\,(r = 3) + pr\,(r = 6) + pr\,(r = 9) + pr\,(r = 12)$$

$$= \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{12}{36} = \frac{1}{3}$$

$\therefore$ choice (b) pr r/3 is an integer = 5/6 is wrong

Consider choice (c)

$$pr\,(r = 8 \mid r/4 \text{ isan integer}) = \frac{1}{36}$$

Now, $$pr\,(r/4 \text{ is an integer}) = pr\,(r = 4) + pr\,(r = 8) + pr\,(r = 12)$$

$$= \frac{3}{36} + \frac{5}{36} + \frac{1}{36} = \frac{9}{36} = \frac{1}{4}$$

$$pr(r = 8 \text{ and } r/4 \text{ is an integer}) = pr\,(r = 8) = \frac{5}{36}$$

$$\therefore \quad pr\,(r = 8 \mid r/4 \text{ is aninteger}) = \frac{5/36}{1/4} = \frac{20}{36} = \frac{5}{9}$$

∴ Choice (c) is correct.

Q.25 A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Give that the first removed ball is white, the probability that the second removed ball is red is

(a) 1/3
(b) 3/7
(c) 1/2
(d) 4/7

[EE, GATE-2010, 2 marks]

Solution: (c)

$$p\,(\text{II is red} \mid \text{I is white}) = \frac{p(\text{II is red and I is white})}{p(\text{I is white})} = \frac{p(\text{I is white and II is red})}{p(\text{I is white})}$$

$$= \frac{\frac{4}{7} \times \frac{3}{6}}{\frac{4}{7}} = \frac{3}{6} = \frac{1}{2}$$

Q.26 Let P (E) denote the probability of the event E. Given P (A) = 1, P(B) = 1/2, the values of P(A/B) and P(B/A) respectively are

(a) 1/4, 1/2
(b) 1/2, 1/4
(c) 1/2, 1
(d) 1, 1/2

[CS, GATE-2003, 1 mark]

Solution: (d)

Given, $$P(A) = 1$$
$$P(B) = 1/2$$

Both events are independent

So, $$P(A \cap B) = 1/2$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/2}{1/2} = 1$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/2}{1} = 1/2$$

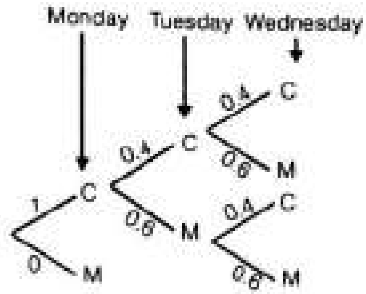
Q.27 Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability that she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability that she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probablity that she studies computer science on Wednesday?

(a) 0.24
(b) 0.36
(c) 0.4
(d) 0.6

[CS, GATE-2008, 2 marks]

Solution: (c)

Let C denote computes science study and M denotes maths study. The tree diagram for the problem can be represented as shown below:



Now by rule of total probability we total up the desired branches and get the answer as shown below:

p(C on monday and C on wednesday)

= p(C on monday, C on tuesday and C on wednesday) + p(C on monday, M on tuesday and C on wednesday)

$$= 1 \times 0.6 \times 0.4 + 1 \times 0.4 \times 0.4 = 0.24 + 0.16 = 0.40$$

Q.28 An unbalanced dice (with 6 faces, numbered from 1 to 6) is thrown. The probability that the face value is odd is 90% of the probability that the face value is even. The probability of getting any even numbered face is the same.

If the probability that the face is even given that it is greater than 3 is 0.75, which one of the following options is closest to the probability that the face value exceeds 3?

(a) 0.453 (b) 0.468
(c) 0.485 (d) 0.492

[CS, GATE-2009, 2 marks]

Solution: (b)

It is given that

$$p(\text{odd}) = 0.9\,p(\text{even})$$

Now since, $\Sigma p(x) = 1$

$\therefore$ $p(\text{odd}) + p(\text{even}) = 1$

$\Rightarrow$ $0.9\,p(\text{even}) + p(\text{even}) = 1$

$\Rightarrow$ $p(\text{even}) = \dfrac{1}{1.9} = 0.5263$

Now, it is given that p (any even face) is same

i.e. $p(2) = p(4) = p(6)$

Now since, $p(\text{even}) = p(2) \text{ or } p(4) \text{ or } p(6) = p(2) + p(4) + p(6)$

$\therefore$ $p(2) = p(4) = p(6) = \dfrac{1}{3}\,p(\text{even}) = \dfrac{1}{3}(0.5263) = 0.1754$

It is given that

$$p(\text{even} \mid \text{face} > 3) = 0.75$$

$\Rightarrow$ $\dfrac{p(\text{even} \cap \text{face} > 3)}{p(\text{face} > 3)} = 0.75$

$\Rightarrow$ $\dfrac{p(\text{face} = 4, 6)}{p(\text{face} > 3)} = 0.75$

$$\Rightarrow \quad p\,(\text{face} > 3) = \frac{p(\text{face} = 4,6)}{0.75} = \frac{p(4)+p(6)}{0.75}$$

$$= \frac{0.1754+0.1754}{0.75} = 0.4677 \simeq 0.468$$

Q.29 If two fair coins are flipped and at least one of the outcomes is known to be a head, what is the probability that both outcomes are heads?

(a) 1/3
(b) 1/4
(c) 1/2
(d) 2/3

[CS, GATE-2009, 1 marks]

Solution: (a)

Sample space = {HT, TH, HH}

Both outcomes head = {HH}

$$\text{Required probability} = \frac{1}{3}$$

Q.30 X and Y are two random independent events. It is known that $P(X) = 0.40$ and $P(X \cup Y^C) = 0.7$. Which one of the following is the value of $P(X \cup Y)$?

(a) 0.7
(b) 0.5
(c) 0.4
(d) 0.3

[CE, 2016 : 1 Mark, Set-II]

Solution: (a)

$P(X \cup Y^c) = 07$

$\Rightarrow P(X) + P(Y^c) - P(X)\,P(Y^c) = 0.7$

(Since X, Y are independent events)

$\Rightarrow P(X) + 1 - P(Y) - P(X)\,[1 - P(Y)] = 0$

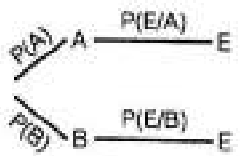
$\Rightarrow P(Y) - P(X \cap Y) = 0.3$ ...(i)

$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= 0.4 + 0.3 = 0.7$

Rule 5: Rule of Total Probability

Consider an event E which occurs via two different events A and B. Further more, let A & B be mutually exclusive & collectively exhaustive events. This situation may be represented by following tree diagram



Now, the probability of E is given by value of total probability as

$$P(E) = P(A \cap E) + P(B \cap E) = P(A) * P(E/A) + P(B) *(E/B)$$

This is called rule of total probability.

Sometimes however, we may wish to know that, given that the event E has already occured, what is the probability that it occurred with A? In this case we can use Bayes Theorem given below.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.31 Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is

(a) $\frac{5}{11}$
(b) $\frac{1}{2}$
(c) $\frac{7}{13}$
(d) $\frac{6}{11}$

[EE, GATE-2015 : 2 Marks, Set-1]

Solution: (d)

$$P(\text{A wins}) = p(6 \text{ in first throw by } A) + p(A \text{ not } 6, B \text{ not } 6, A\, 6) + \ldots$$

$$= \frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \cdots$$

$$= \frac{1}{6}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right) = \frac{1}{6}\cdot\frac{1}{1-\left(\frac{5}{6}\right)^2} = \frac{6}{11}$$

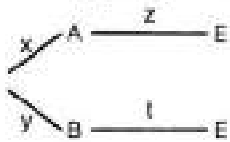
Rule 6: Bayes Theorem

i.e.
$$P(A|E) = \frac{P(A\cap E)}{P(E)} = \frac{P(A\cap E)}{P(A\cap E)+P(B\cap E)}$$

$$= \frac{P(A)*P(E|A)}{P(A)*P(E|A)+P(B)*P(E|B)}$$

This formula is called Baye's Theorem. Notice that the denominator of Bayes theorem or formula is obtained by using the rule of total probability.

If the tree diagram for the problem is more simply represented as



Then the above Baye's formula gives

$$p(A|E) = \frac{xz}{xz + yt}$$

and
$$p(B|E) = \frac{yt}{xz + yt}$$

ILLUSTRATIVE EXAMPLES

Example:

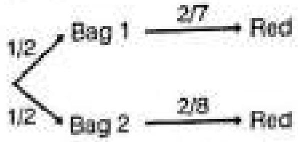
Suppose we have 2 bags. Bag 1 contains 2 red & 5 green marbles. Bag 2 contains 2 red and 6 green marbles. A person tosses a coin & if it is heads goes to bag 1 and draws a marble. If it is tails, he goes to bag 2 and draws a marble. In this situation.

(a) What is the probability that the marble drawn this is Red?

(b) Given that the marble draw is red, what is probability that it came from bag 1.

Solution:

The tree diagram for above problem, is shown below:



(a) $\therefore$ $P(\text{Red}) = 1/2 \times 2/7 + 1/2 \times 2/8$

(b) $$P(\text{bag1}|\text{Red}) = \frac{P(\text{bag1} \cap \text{Red})}{P(\text{Red})} = \frac{1/2 \times 2/7}{1/2 \times 2/7 + 1/2 \times 2/8}$$

$$= \frac{1/7}{15/56} = 8/15$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.32 A fair (unbiased) coin was tossed four times in succession and resulted in the following outcomes: (i) Head, (ii) Head, (iii) Head, (iv) Head. The probability of obtaining a 'Tail' when the coin is tossed again is

(a) 0
(b) $\frac{1}{2}$
(c) $\frac{4}{5}$
(d) $\frac{1}{5}$

[CE, GATE-2014 : 1 Mark, Set-2]

Solution : (b)

$$P(E) = \frac{n(E)}{n(S)}$$

$$n(s) = [\{H\}, \{T\}] = 2$$

$$n(E) = \{(T)\} = 1$$

$$\therefore \quad P(E) = \frac{1}{2}$$

Q.33 A box contains 25 parts of which 10 are defective. Two parts are being drawn simultaneously in a random manner from the box. The probability of both the parts being good is

(a) $\frac{7}{20}$
(b) $\frac{45}{125}$
(c) $\frac{25}{29}$
(d) $\frac{5}{9}$

[ME, GATE-2014 : 1 Mark, Set-2]

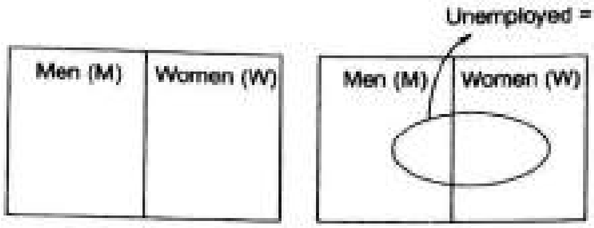
Solution : (a)

$$\text{required prob} = \frac{{}^{15}C_2}{{}^{25}C_2} = \frac{14 \times 15}{25 \times 24} = \frac{7}{20}$$

Q.34 A group consists of equal number of men and women. Of this group 20% of the men and 50% of the women are unemployed. If a person is selected at random from this group, the probability of the selected person being employed is _____.

[ME, GATE-2014 : 1 Mark, Set-3]

Solution :



$$P(M) = \frac{1}{2} \qquad P(U|M) = 0.2$$

$$P(W) = \frac{1}{2} \qquad P(U|W) = 0.5$$

Let E = Employed person

$$P(E|M) = 1 - 0.2 = 0.8 \quad ; \quad P(E|W) = 1 - 0.5 = 0.5$$

By total probability

Probability of selecting employed person,

$$P(E) = P(M) \cdot P(E|M) + P(W) \cdot P(E|W)$$

$$= \frac{1}{2} \times 0.8 + \frac{1}{2} \times 0.5 = 0.65$$

Q.35 A fair coin is tossed n times. The probability that the difference between the number of heads and tails is (n – 3) is

(a) 2^{-n}
(b) 0
(c) ${}^{n}C_{n-3}\, 2^{-n}$
(d) 2^{-n+3}

[EE, GATE-2014 : 2 Marks, Set-1]

Solution : (b)

Let number of heads = x. So number of tails will be n – x. We want the difference between the number of heads and number of tails to be n – 3

i.e. $x - (n - x) = n - 3.$

$\Rightarrow \quad x = \frac{2n-3}{2} = n - \frac{3}{2}$ which is not an integer

∴ which is an impossible event so, the required probability is zero.

Q.36 Consider a dice with the property that the probability of a face with n dots showing up is proportional to n. The probability of the face with three dots showing up is ____.

[EE, GATE-2014 : 1 Mark, Set-2]

Solution :

Let probability of occurence of one dot is P.

So, writing total probability

$P + 2P + 3P + 4P + 5P + 6P = 1$

$$P = \frac{1}{21}$$

hence problem of occurence of 3 dot is

$$= 3P = \frac{3}{21} = \frac{1}{7} = 0.142$$

Q.37 Three companies X, Y and Z supply computers to a university. The percentage of computers supplied by them and the probability of those being defective are tabulated below:

Company	% of computer supplied	Probability of being defective
X	60%	0.01
Y	30%	0.02
Z	10%	0.03

Given that a computer is defective, the probability that it was supplied by Y is

(a) 0.1 (b) 0.2

(c) 0.3 (d) 0.4 **[EC, GATE-2006, 2 marks]**

Solution: (d)

S → supply by y, d → defective

Probability that the computer was supplied by y, if the product is defective

$$P(s/d) = \frac{P(s \cap d)}{P(d)}$$

$$P(s \cap d) = 0.3 \times 0.02 = 0.006$$

$$P(d) = 0.6 \times 0.1 + 0.3 \times 0.02 + 0.1 \times 0.03 = 0.015$$

$$P(s/d) = \frac{0.006}{0.015} = 0.4$$

Q.38 In a housing society, half of the families have a single child per family, while the remaining half have two children per family. The probability that a child picked at random, has a sibling is **[EC, GATE-2014 : 1 Mark, Set-1]**

Solution :

Let there n families. Now $\frac{n}{2}$ families have single child and $\frac{n}{2}$ families have two children. So total number of children is

$$= \frac{n}{2} \times 1 + \frac{n}{2} \times 2 = \frac{3n}{2}$$

Now, favourable case is the child picked at random has sibling = n.

So probability (a child picked at random, has a sibling) = $\frac{n}{\frac{3n}{2}} = \frac{2}{3} = 0.666$

Q.39 An unbiased coin is tossed an infinite number of times. The probability that the fourth head appears at the tenth toss is

(a) 0.067 (b) 0.073

(c) 0.082 (d) 0.091 **[EC, GATE-2014 : 1 Mark, Set-3]**

Solution : (c)

It means 3-head appears in 1st 9 trials.

Probability of getting exactly 3 head in 1st 9 trials

$$= {}^9C_3 \times \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^6 = {}^9C_3 \times \left(\frac{1}{2}\right)^9$$

and in 10th trial head must appears.

So required probability

$$= {}^9C_3 \left(\frac{1}{2}\right)^9 \times \frac{1}{2} = \frac{84}{1024} = 0.082$$

Q.40 Parcels from sender S to receiver R pass sequentially through two post-offices. Each post-office has a probability 1/5 of losing an incoming parcel, independently of all other parcels. Given that a parcel is lost, the probability that it was lost by the second post-office is ________.

Solution : **[EC, GATE-2014 : 2 Marks, Set-4]**

$$\text{Probability to lost at port-office 1} = \frac{1}{5}$$

$$\text{Probability to lost at port-office 2} = \frac{4}{5} \times \frac{1}{5}$$

$$\text{Total probability to lost} = \frac{1}{5} + \frac{4}{25} = \frac{9}{25}$$

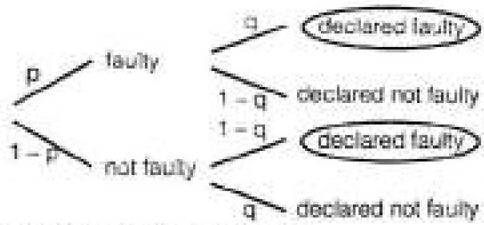
$$\text{Required probability} = \frac{4/25}{9/25} = \frac{4}{9} = 0.444$$

Q.41 Consider a company that assembles computers. The probability of a faulty assembly of any computer is p. The company therefore subjects each computer to a testing process. This testing process gives the correct result for any computer with a probability of q. What is the probability of a computer being declared faulty?

(a) $pq + (1 - p)(1 - q)$
(b) $(1 - q)p$
(c) $(1 - p)q$
(d) pq

[CS, GATE-2010, 2 marks]

Solution: (a)



The tree diagram of probabilities is shown above.
From above tree, by rule of total probability,

$$p(\text{declared faulty}) = pq + (1 - p)(1 - q)$$

Q.42 Four fair six-sided dice are rolled. The probability that the sum of the results being 22 is X/1296. The value of X is ______.

[CS, GATE-2014 : 2 Marks, Set-1]

Solution: (10)

$$6, 6, 6, 4 \Rightarrow \frac{4!}{3!} = 4 \text{ ways}$$

$$6, 6, 5, 5 \Rightarrow \frac{4!}{2!\,2!} = 6 \text{ ways}$$

$$\text{Probability of sum to be } 22 = \frac{6+4}{6^4} = \frac{6+4}{1296} = \frac{x}{1296}$$

$$\Rightarrow \quad x = 10$$

Q.43 The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2, 3 or 5 is ______.

[CS, GATE-2014 : 2 Marks- Set-2]

Solution: (0.26)

$$1 \le x \le 100$$

P(x is not divisible by 2, 3 or 5) = 1 – P(x is divisible by 2, 3 or 5)

$$= 1-\left[\left[\frac{100}{2}\right]+\left[\frac{100}{3}\right]+\left[\frac{100}{5}\right]-\left[\frac{100}{6}\right]-\left[\frac{100}{15}\right]-\left[\frac{100}{10}\right]+\left[\frac{100}{30}\right]\right]$$

$$1-\frac{74}{100} = 0.26$$

Q.44 Let S be a sample space and two mutually exclusive events A and B be such that $A \cup B = S$. If P(·) denotes the probability of the event, the maximum value of P(A) P(B) is ______.

[CS, 2014 : 2 Marks-Set-3]

Solution: (0.25)

It is given that A and B are mutually exclusive also it is given that $A \cup B = S$

which means that A and B are collectively exhaustive.

Now if two events A and B are both mutually exclusive and collectively exhaustive, then

$P(A) + P(B) = 1 \Rightarrow P(B) = 1 - P(A)$

Now we wish to maximize P(A) P(B)

$$= P(A)(1 - P(A))$$

Let $\quad P(A) = x$

Now $\quad P(A)(1 - P(A)) = x(1 - x) = x - x^2$

Say $\quad y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x = 0 \Rightarrow x = \frac{1}{2}$$

$$= \frac{d^2y}{dx^2} = -2 < 0;\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{2}} = -2 < 0$$

y has maximum at $x = 1/2$, $y_{max} = \frac{1}{2}-\left(\frac{1}{2}\right)^2 = 0.25$

Q.45 Consider the following experiment.

Step 1. Flip a fair coin twice.

Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.

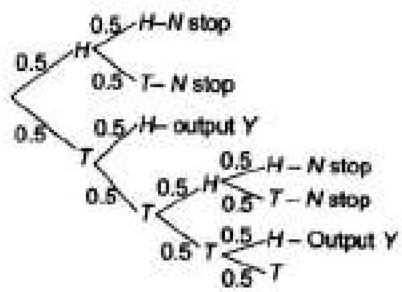
Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.

Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.

The probability that the output of the experiment is Y is ______ (up to two decimal places).

[CS, 2016 : 2 Marks, Set-1]

Solution:



The tree diagram for the problem is given above.
The desired output is Y.
Now by rule of total probability

$p(\text{output} = Y) = 0.5 \times 0.5 + 0.5 \times 0.5 \times 0.5 \times 0.5 + ...$

Infinite geometric series with

$$a = 0.5 \times 0.5$$

and $$r = 0.5 \times 0.5$$

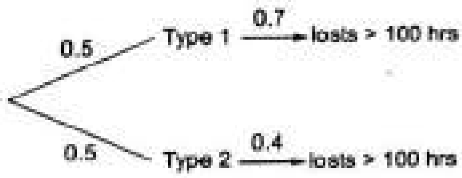
so $$p(\text{output} = Y) = \frac{0.5 \times 0.5}{1 - 0.5 \times 0.5} = \frac{0.25}{0.75}$$

$$\frac{1}{3} = 0.33 \text{ (upto 2 decimal places)}$$

Q.46 Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is______.

[CS, 2016 : 1 Mark, Set-2]

Solution:



$$P(\text{losts} > 100\text{ hr}) = 0.5 \times 0.7 + 0.5 \times 0.4$$
$$= 0.35 + 0.2 = 0.55$$

Q.47 Three cards were drawn from a pack of 52 cards. The probability that they are a king, a queen, and a jack is

(a) $\frac{16}{5525}$

(b) $\frac{64}{2197}$

(c) $\frac{3}{13}$

(d) $\frac{8}{16575}$

[ME, 2016 : 2 Marks, Set-3]

Solution: (a)

$$\frac{4_{C_1} \cdot 4_{C_1} \cdot 4_{C_1}}{52_{C_3}} = \frac{64}{\frac{52 \times 51 \times 50}{3 \times 2 \times 1}} = \frac{64}{22,100} = \frac{16}{5525}$$

5.2 STATISTICS

5.2.1 Introduction

Statistics is a branch of mathematics which gives us the tools to deal with large quantities of data and derive meaningful conclusions about the data. To do this, statistics uses some numbers or measures which describe the general features contained in the data. In other words, using statistics, we can summarise large quantities of data, by a few descriptive measures.

Two descriptive measures are often used to summarise data sets. These are

1. Measure of central tendency
2. Measure of dispersion

The central tendency measure indicates the average value of data, where "average" is a generic term used to indicate a representative value that describes the general centre of the data.

The dispersion measure characterises the extent to which data items differ from the central tendency value. In other words dispersion measures and quantifies the variation in data. The larger this number, the more the variation amongst the data items.

Mean, Median and Mode are some examples of central tendency measures.

Standard deviation, variance and coefficient of variation are examples of dispersion measures.

Now we will study each of these six statistical measures in greater detail.

5.2.2 Arithmetic Mean

5.2.2.1 Arithmatic Mean for Raw Data

The formula for calculating the arithmetic mean for raw data is: $\bar{x} = \frac{\Sigma x}{n}$

$\bar{x}$ - arithmetic mean

x - refers to the value of an observation

n - number of observations.

ILLUSTRATIVE EXAMPLES

Example:

The number of visits made by ten mothers to a clinic were; 8 6 5 5 7 4 5 9 7 4

Calculate the average number of visits.

Solution:

Σx = total of all these numbers of visits, that is the total number of visits made by all mothers.

$8 + 6 + 5 + 5 + 7 + 4 + 5 + 9 + 7 + 4 = 60$

Number of mothers $n = 10$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{60}{10} = 6$$

5.2.2.2 The Arithmatic Mean for Grouped Data (Frequency Distribution)

The formula for the arithmetic mean calculated from a frequency distribution has to be amended to include the frequency. It becomes

$$\bar{x} = \frac{\Sigma(fx)}{\Sigma f}$$

ILLUSTRATIVE EXAMPLES

Example:

To show how we can calculate the arithmetic mean of a grouped frequency distribution, there is a example of weights of 75 pigs.

The classes and frequencies are given in following table.

Weight (kg)	Midpoint of class x	Number of pigs f (frequency)	fx
0 & under 20	15	1	15
20 & under 30	25	7	175
30 & under 40	35	8	280
40 & under 40	45	11	495
50 & under 60	55	19	1045
60 & under 70	65	10	650
70 & under 80	75	7	525
80 & under 90	85	5	425
90 & under 100	95	4	380
100 & under 110	105	3	215
Total		75	4305

Solution:

With such a frequency distribution we have a range of values of the variable comprising each group. As our values for x in the formula for the arithmetic mean we use the midpoints of the classes.

In this case $$\bar{x} = \frac{\Sigma(fx)}{\Sigma f} = \frac{4305}{75} = 57.4 \text{ kg}$$

5.2.3 Median

Arithmetic mean is the central value of the distribution in the sense that positive and negative deviations from the arithmetic mean balance each other. It is a quantitative average.

On the other hand, **median** is the central value of the distribution in the sense that the number of values less than the median is equal to the number of values greater than the median. So, median is a positional average. Median is the central value in a sense different from the arithmetic mean. In case of the arithmetic mean it is the **"numerical magnitude"** of the deviations that balances. But, for the median it is the 'number of values greater than the median which balances against the number of values of less than the median.

5.2.3.1 Median for Raw Data

In general, if we have n values of x, they can be arranged in ascending order as:

$$x_1 < x_2 < \ldots\ldots < x_n$$

Suppose n is odd, then Median = the $\frac{(n+1)}{2}$-th value

However, if n is even, we have two middle points

$$\text{Median} = \frac{\left(\frac{n}{2}\right)^{th} \text{value} + \left(\frac{n}{2}+1\right)^{th} \text{value}}{2}$$

ILLUSTRATIVE EXAMPLES

Example:

The heights (in cm) of six students in class are 160, 157, 156, 161, 159, 162. What is median height?

Solution:

Arranging the heights in ascending order 156, 157, 159, 160, 161, 162

Two middle most values are the 3rd and 4th.

$$\text{Median} = \frac{1}{2}(159 + 160) = 159.5$$

5.2.3.2 Median for Grouped Data

1. Identify the median class which contains the middle observation $\left(\frac{N+1}{2}\right)^{th}$ observation) This can be done by observing the first class in which the cumulation frequency is equal to or more than $\frac{N+1}{2}$. Here, $N = \Sigma f$ = total number of observations.
2. Calculate Median as follows:

$$\text{Median} = L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right] \times h$$

Where,
- L = Lower limit of median class
- N = Total number of data items = ΣF
- F = Cumulative frequency of the class immediately preceding the median class
- f_m = Frequency of median class
- h = width of median class

ILLUSTRATIVE EXAMPLES

Example:

Consider the following table giving the marks obtained by students in an exam

MarkRange	f No of Students	Cumulative Frequency
0 – 20	2	2
20 – 40	3	5
40 – 60	10	15
60 – 80	15	30
80 – 100	20	50

Solution:

Here, $$\frac{N+1}{2} = 25.5$$

The class 60-80 is the median class since cumulative frequency is 30 >25.5

$$\text{Median} = \frac{60 + [25.5 - (15+1)]}{15} \times 20 = 69.66 \simeq 69.7$$

∴ Median marks of the class is approximately 69.7.

i.e. (at least) half the students got less than 69.7 and (almost) half got more than 69.7 marks.

5.2.4 Mode

Mode is defined as the value of the variable which occurs most frequently.

5.2.4.1 Mode for Raw Data

In raw data, the most frequently occuring observation is the mode. That is data with highest frequency is mode. If there is more than one data with highest frequency, then each of them is a mode. Thus we have Unimodal (single mode), Bimodal (two modes) and Trimodal (three modes) data sets.

ILLUSTRATIVE EXAMPLES

Example:

Find the mode of the data set: 50, 50, 70, 50, 50, 70, 60.

Solution:

1. Arrange in ascending order: 50, 50, 50, 50, 60, 70, 70
2. Make a discrete data frequency table:

Data	Frequency
50	4
60	1
70	2

Since, 50 is the data with maximum frequency, mode is 50. This is a unimodal data set.

5.2.4.2 Mode for Grouped Data

Mode is that value of x for which the frequency is maximum. If the values of x are grouped into the classes (such that they are uniformly distributed within any class) and we have a frequency distribution then:

1. Identify the class which has the largest frequency (modal class)
2. Calculate the mode as

$$\text{Mode} = L + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

Where,
L = Lower limit of the modal class
f_0 = Largest frequency (frequency of Modal Class)
f_1 = Frequency in the class preceding the modal class
f_2 = Frequency in the class next to the modal class
h = Width of the modal class

ILLUSTRATIVE EXAMPLES

Example:

Data relating to the height of 352 school students are given in the following frequency distribution. Calculate the modal height.

Heigh (in feet)	Number of students
3.0 – 3.5	12
3.5 – 4.0	37
4.0 – 4.5	79
4.5 – 5.0	152
5.0 – 5.5	65
5.5 – 6.0	7
Total	352

Solution:

Since, 152 is the largest frequency, the modal class is (4.5 – 5.0)

Thus, $L = 4.5$, $f_0 = 152$, $f_1 = 79$, $f_2 = 65$, $h = 0.5$.

$$\text{Mode} = 4.5 + \frac{152 - 79}{2(152) - 79 - 65} \times 0.5 = 4.73 \text{ (approx.)}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.48 Which one of the following statements is NOT true?

(a) The measure of skewness is dependent upon the amount of dispersion

(b) In a symmetric distribution, the values of mean, mode and median are the same

(c) In a positively skewed distribution: mean > median > mode

(d) In a negatively skewed distribution: mode > mean > median

[CE, GATE-2005, 1 mark]

Solution: (d)

a, b, c are true but (d) is not true since in a negatively skewed distribution, mode > median > mean.

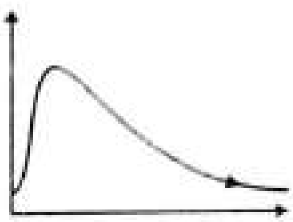
5.2.5 Properties Relating Mean, Median and Mode

1. Empirical mode = 3 median – 2 mean

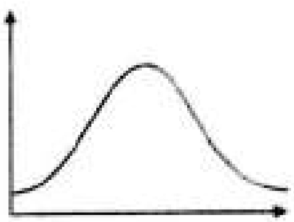
   when an approximate value of mode is required above empirical formula for mode may be used.

2. There are three types of frequency distributions.

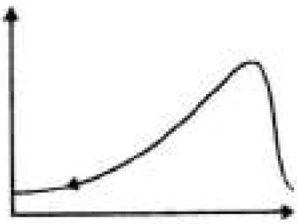
   Positively skewed, symmetric and negatively skewed distribution.



(a) Positively Skewed



(a) Symmetric



(a) Negatively Skewed

(a) In positively skewed distribution:

$$\text{Mode} \le \text{Median} \le \text{Mean}$$

(b) In symmetric distribution:

$$\text{Mean} = \text{Median} = \text{Mode}$$

(c) In negatively skewed distribution:

$$\text{Mean} \le \text{Median} \le \text{Mode}$$

5.2.6 Standard Deviation

Standard Deviation is a measure of dispersion or variation amongst data.

Instead of taking absolute deviation from the arithmetic mean, we may square each deviation and obtain the arithmetic mean of squared deviations. This gives us the 'variance' of the values.

The positive square root of the variance is called the **'Standard Deviation'** of the given values.

5.2.6.1 Standard Deviation for Raw Data

Suppose $x_1, x_2, \ldots, x_n$ are n values of the x, their arithmetic mean is:

$\bar{x} = \frac{1}{N}\Sigma x_i$ and $x_1 - \bar{x}, x_2 - \bar{x} \dots x_n - \bar{x}$ are the deviations of the values of x from $\bar{x}$. Then

$\sigma^2 = \frac{1}{n^2}\Sigma(x_i - \bar{x})^2$ is the variance of x. It can be shown that

$$\sigma^2 = \frac{\Sigma(x_i - \bar{x})^2}{n} = \frac{1}{n}\Sigma x_i^2 - \bar{x}^2 = \frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{n^2}$$

It is conventional to represents the variance by the symbol σ^2. Infact, σ is small sigma and Σ is capital sigma.

Square root of the variance is the standard deviation

$$\sigma = +\sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2} = \sqrt{\tfrac{1}{n}\Sigma x_i^2 - \bar{x}^2} = \sqrt{\frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{n^2}}$$

ILLUSTRATIVE EXAMPLES

Example:

Consider three students in a class, and their marks in exam was 50, 60 and 70. What is the standard deviation of this data set?

Solution:

Student	x_i Marks	x_i^2
A	50	2500
B	60	3600
C	70	4900
	180	11000

Here, $n = 3$

$$\text{Standard Deviation } (\sigma) = \sqrt{\frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{n^2}} = \sqrt{\frac{3 \times 11000 - (180)^2}{3^2}}$$

$$= 8.165$$

$$\text{Variance} = \sigma^2 = 66.67$$

5.2.6.2 Standard Deviation for Grouped Data

Calculation for standard deviation for grouped data can be shown by this example:

ILLUSTRATIVE EXAMPLES

Example:

The frequency distribution for heights of 150 young ladies in a beauty contest is given below for which we have to calculate standard deviation.

Solution:

Height (in inches)	Mid values x	Frequency f	f × x	f × x^2
62.0 – 63.5	62.75	12	753.00	47250.75
63.5 – 65.0	64.25	20	1285.00	82561.25
65.0 – 66.5	65.75	28	1841.00	121045.75
66.5 – 68.0	67.25	18	1210.50	81406.125
68.0 – 69.5	68.75	19	1306.25	89806.125
69.5 – 71.0	70.25	20	1405.00	89804.6875
71.0 – 72.5	71.75	30	2152.50	98701.25
72.5 – 74.0	73.25	3	219.75	154441.875
				16096.6875
Total		150	10173.00	691308.375

Thus, $\bar{x} = \frac{\Sigma f x_i}{\Sigma f} = \frac{10173}{150} = 67.82$

where, $N = \Sigma f_i = 150$

Therefore, the standard deviation of x is

$$= \sigma_x = \sqrt{\frac{\Sigma f x_i^2}{N} - \bar{x}^2}$$

$$= \sqrt{\frac{N\Sigma f x_i^2 - (\Sigma f x_i)^2}{N^2}} = \sqrt{\frac{150 \times 691308.375 - (10173)^2}{(150)^2}}$$

$$= 3.03$$

$$\text{Variance} = \sigma_x^2 = (3.03)^2 = 9.170$$

5.2.7 Variance

The square of standard deviation (σ) is called as the variance (σ^2).

So if $\sigma = 10$, then variance $= \sigma^2 = 100$.

Alternatively if variance $= \sigma^2 = 100$ then standard deviation $= \sqrt{\text{Variance}} = \sqrt{100} = 10$

The larger the standard deviation, larger will be the variance.

5.2.8 Coefficient of Variation

The standard deviation is an absolute measure of dispersion and hence can not be used for comparing variability of 2 data sets with different means.

Therefore, such comparisons are done by using a relative measure of dispersion called coefficient of variation (CV).

$$CV = \frac{\sigma}{\mu}$$

where σ is the standard deviation and μ is the mean of the data set.

CV is often represented as a percentage,

$$CV\,\% = \frac{\sigma}{\mu} \times 100$$

When comparing data sets, the data set with larger value of CV% is more variable (less consistent) as compared to a data set with lesser value of CV%.

For example:

	μ	σ	CV %
Data set 1	5	1	20%
Data set 2	20	2	10%

Although $\sigma = 2$ for data set 2 is more than $\sigma = 1$ for data set 1, data set 2 is actually less variable compared to data set 1, as can be seen by the fact that data set 2 has a CV % of 10%, while data set 1 has a CV % of 20%.

So comparison of variability between 2 or more data sets (with different means) should be done by comparing CV % and not by comparing standard deviations.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.49 If the standard deviation of the spot speed of vehicles in a highway is 8.8 kmph and the mean speed of the vehicles is 33 kmph, the coefficient of variation in speed is

(a) 0.1517
(b) 0.1867
(c) 0.2666
(d) 0.3646

[CE, GATE-2007, 2 marks]

Solution: (c)

$$CV = \frac{\sigma}{\mu} = \frac{8.8}{33} = 0.2666$$

Q.50 Consider the finite sequence of random values $X = [x_1, x_2, ..., x_n]$. Let μ_x be the mean and σ_x be the standard deviation of X. Let another finite sequence Y of equal length be derived from this as $y_i = a * x_i + b$, where a and b are positive constant. Let μ_y be the mean and σ_y be the standard deviation of this sequence. Which one of the following statements INCORRECT?

(a) Index position of mode of X in X is the same as the index position of mode of Y in Y.
(b) Index position of median of X in X is the same as the index position of median of Y in Y
(c) $\mu_y = a\mu_x + b$
(d) $\sigma_y = a\sigma_x + b$

[CS, GATE-2011, 2 marks]

Solution: (d)

Standard deviation is affected by scale but not by shift of origin.

So $\quad y_i = ax_i + b$

$\Rightarrow \quad \sigma_y = a\sigma_x$

(if a could be negative then $\sigma_y = |a|\sigma_x$ is more correct since standard deviation cannot be negative)

Clearly, $\quad \sigma_y = a\sigma_x + b$ is false

So (d) is incorrect.

Q.51 Type II error in hypothesis testing is

(a) acceptance of the null hypothesis when it is false and should be rejected
(b) rejection of the null hypothesis when it is true and should be accepted
(c) rejection of the null hypothesis when it is false and should be rejected
(d) acceptance of the null hypothesis when it is true and should be accepted

[CE, 2016 : 1 Mark, Set-I]

Solution:

$$f(x) = \begin{cases} \frac{x}{4}(4-x^2), & 0 \le x \le 2 \\ 0 & , \text{Otherwise} \end{cases}$$

$$\mu_x = \int_0^2 x f(x)dx$$

$$\therefore \text{Mean } (\mu_x) = \int_0^2 x \frac{x}{4}(4-x^2)dx$$

$$= \int_0^2 \left(x^2 - \frac{x^4}{4}\right)dx = \left[\frac{x^3}{3} - \frac{x^5}{20}\right]_0^2 = \frac{8}{3} - \frac{32}{20} = \frac{16}{15} = 1.066$$

Q.52 The spot speeds (expressed in km/hr) observed at a road section are 66, 62, 45, 79, 32, 51, 56, 60, 53, and 49. The median speed (expressed in km/hr) is ______.
(**Note:** answer with one decimal accuracy)

[CE, 2016 : 1 Mark, Set-II]

Solution:

Median speed is the speed at the middle value in series of spot speeds that are arranged in ascending order. 50% of speed values will be greater than the median 50% will be less than the median.

Ascending order of spot speed studies are

32, 39, 45, 51, 53, 56, 60, 62, 66, 79

$$\text{Median speed} = \frac{53+56}{2} = 54.5 \text{ km/hr}$$

Q.53 If $f(x)$ and $g(x)$ are two probability density functions,

$$f(x) = \begin{cases} \frac{x}{a}+1 & : -a \le x < 0 \\ -\frac{x}{a}+1 & : 0 \le x \le a \\ 0 & : \text{otherwise} \end{cases} \quad ; \quad g(x) = \begin{cases} -\frac{x}{a} & : -a \le x < 0 \\ \frac{x}{a} & : 0 \le x \le a \\ 0 & : \text{otherwise} \end{cases}$$

Which one of the following statements is true?

(a) Mean of $f(x)$ and $g(x)$ are same; Variance of $f(x)$ and $g(x)$ are same
(b) Mean of $f(x)$ and $g(x)$ are same; Variance of $f(x)$ and $g(x)$ are different
(c) Mean of $f(x)$ and $g(x)$ are different; Variance of $f(x)$ and $g(x)$ are same
(d) Mean of $f(x)$ and $g(x)$ are different; Variance of $f(x)$ and $g(x)$ are different

[CE, 2016 : 2 Marks, Set-II]

Solution: (b)

Mean of f(x) is E(x)

$$= \int_{-a}^{0} x\left(\frac{x}{a}+1\right)dx + \int_0^a x\left(\frac{-x}{a}+1\right)dx$$

$$= \left(\frac{x^3}{3a}+\frac{x^2}{2}\right)_{-a}^{0} + \left(\frac{-x^3}{3a}+\frac{x^2}{2}\right)_0^a = 0$$

Variance of f(x) is $E(x)^2 - \{E(x)^2\}$ where

$$E(x)^2 = \int_{-a}^{0} x^2\left(\frac{x}{a}+1\right)dx + \int_{0}^{a} x^2\left(\frac{-x}{a}+1\right)dx$$

$$= \left(\frac{x^4}{4a}+\frac{x^3}{3}\right)_{-a}^{0} + \left(\frac{-x^4}{4a}+\frac{x^3}{3}\right)_{0}^{a} = \frac{a^3}{6}$$

$\Rightarrow$ Variance is $\frac{a^3}{6}$

Next, mean of g(x) is E(x)

$$= \int_{a}^{0} x\left(\frac{-x}{a}\right)dx + \int_{0}^{a} x\left(\frac{x}{a}\right)dx = 0$$

Variance of g(x) is $E(x^2) - \{E(x)\}^2$, where

$$E(x^2) = \int_{-a}^{0} x^2\left(\frac{-x}{a}\right)dx + \int_{0}^{a} x^2\left(\frac{x}{a}\right)dx = \frac{a^3}{2}$$

$\Rightarrow$ Variance is $\frac{a^3}{2}$

$\therefore$ Mean of f(x) and g(x) are same but variance of f(x) and g9x) are different.

5.3 PROBABILITY DISTRIBUTIONS

5.3.1 Random Variables

It is frequently the case when an experiment is performed that we are mainly interested in some function of the outcome as opposed to the actual outcome itself.

For instances, in tossing dice we are often interested in the sum of two dice and are not really concerned about the separate value of each die. That is, we may be interested in knowing that the sum is 7 and not be concerned over whether the actual outcome was (1, 6) or (2, 5) or (3, 4) or (4, 3) or (5, 2) or (6, 1).

Also, in coin flipping we may be interested in the total number of heads that occur and not care at all about the actual head tail sequence that results. These quantities of interest, or more formally, these real valued functions defined on the sample space, are known as random variables.

Because the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

Types of Random Variable: Random variable may be discrete or continuous.

Discrete Random Variable: A variable that can take one value from a discrete set of values.

Example: Let x denotes sum of 2 dice, Now x is a discrete random variable as it can take one value from the set {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, since the sum of 2 dice can only be one of these values.

Continuous Random Variable: A variable that can take one value form a continuous range of values.

Example: x denotes the volume of Pepsi in a 500ml cup. Now x may be a number from 0 to 500, any of which value, x may take.

5.3.1.1 Probability density function (PDF)

Let x be continuous random variable then its PDF F(x) is defined such that

1. $F(x) \geq 0$ 2. $\int_{-\infty}^{\infty} F(x)dx = 1$ 3. $P(a < x < b) = \int_{a}^{b} F(x)dx$

5.3.1.2 Probability mass function (PMF)

Let x be discrete random variable then its PMF p(x) is defined such that

1. $p(x) = P[X = x]$ 2. $p(x) \geq 0$ 3. $\Sigma p(x) = 1$

5.3.2 Distributions

Based on this we can divide distributions also into **discrete distribution** (based on a discrete random variable) or **continuous distribution** (based on a continuous random variable).

Examples of discrete distribution are binomial, Poisson and hypergeometric distributions.

Examples of continuous distribution are uniform, normal and exponential distributions.

5.3.2.1 Properties of Discrete Distribution

$$\Sigma P(x) = 1$$
$$E(x) = \Sigma x P(x)$$
$$V(x) = E(x^2) - (E(x))^2 = \Sigma x^2 P(x) - [\Sigma x P(x)]^2$$

E(x) denotes expected value or average value of the random variable x, while V(x) denotes the variance of the random variable x.

5.3.2.2 Properties of Continuous Distribution

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$F(x) = \int_{-\infty}^{x} f(x)dx \text{ (cumulative distribution function)}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)dx$$

$$V(x) = E(x^2) - [E(x)]^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} x f(x)dx\right]^2$$

$$p(a < x < b) = p(a \leq x < b) = P(a < x \leq b) = P(a \leq x \leq b) = \int_a^b f(x)dx$$

5.3.3 Types of Distributions

Discrete Distributions:

1. General Discrete Distribution
2. Binomial Distribution
3. Hypergeometric Distribution
4. Geometric Distribution
5. Poisson Distribution

5.3.3.1 General Discrete Distribution

Let X be a discrete random variable.

A table of possible values of *x* versus corresponding probability values p(x) is called as its probability distribution table.

Example:

Let X be the number which comes on a single throw of a dice.

Then probability distribution table of X is given by

X	1	2	3	4	5	6
p(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

In this case p(X) is same for all values of X, but this is not necessary, as following example shows. For example, let X be the sum of the numbers coming on a pair of dice thrown.

Now the probability distribution table can be constructed as follows

x	2	3	4	.....	10	11	12
p(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$		$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Notice, that here p(X) is not same for all values of X.

In any probability distribution table

$$\Sigma p(x) = 1 \text{ is always true}$$

Take the case of simple dice

x	1	2	3	4	5	6
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Notice that $$\Sigma p(x) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = 1$$

From above table, we can compute the following:

$$p(X = 3) = \frac{1}{6}$$

$$p(X \geq 3) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$p(X \leq 3) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$p(X < 4) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

Also from above table, we can compute the expected value and variance of x.

$$E(x) = \Sigma x\, p(x)$$

$$V(x) = E(x^2) - [E(x)]^2 = \Sigma x^2 p(x) - [\Sigma x\, p(x)]^2$$

E(x) is the expected value of x and is similar to an average value of x after infinite number of trials. So, E(x) is sometimes also written as μ_x.

V(x) represents the variability of X. So it is sometimes written as σ_x^2.

So, $\sigma_x = \sqrt{V(x)}$, which is the standard deviation of X.

Also expected value of any function g(x) of x can be computed as follows:

$$E(g(x)) = \Sigma g(x) p(x)$$

For example,

$$E(x^3) = \Sigma x^3\, p(x) \text{ and } E(x^2 + 1) = \Sigma(x^2 + 1)\, p(x)$$

For the single dice probability distribution table,

$$P_x = E(x) = \Sigma x p(x) = 1\times\frac{1}{6}+2\times\frac{1}{6}+.....+6\times\frac{1}{6} = 3.5$$

and
$$\sigma_x^2 = V(x) = \Sigma x^2 p(x) - [\Sigma x\, p(x)]^2$$

$$= \left[1^2\times\frac{1}{6}+2^2\times\frac{1}{6}+3^2\times\frac{1}{6}....+6^2\times\frac{1}{6}\right] - (3.5)^2 = 2.917$$

$\therefore$
$$\sigma_x = \sqrt{2.917} = 1.7078$$

Properties of Expectation and Variance:

If x_1 and x_2 are two random variance and a and b are constants,

$$E(ax_1 + b) = a\,E(x_1) + b \quad \text{...(i)}$$
$$V(ax_1 + b) = a^2\,V(x_1) \quad \text{...(ii)}$$
$$E(ax_1 + bx_2) = a\,E(x_1) + b\,E(x_2) \quad \text{...(iii)}$$
$$V(ax_1 + bx_2) = a^2\,V(x_1) + b^2V(x_2) + 2ab\,\text{cov}(x_1, x_2) \quad \text{...(iv)}$$

where $\text{cov}(x_1, x_2)$ represents the covariance between x_1 and x_2

If x_1 and x_2 are independent, then $\text{cov}(x_1, x_2) = 0$ and the above formula reduces to

$$V(ax_1 + bx_2) = a^2V(x_1) + b^2V(x_2) \quad \text{...(v)}$$

For example, from above formula we can say

$$E(x_1 + x_2) = E(x_1) + E(x_2)$$
$$E(x_1 - x_2) = E(x_1) - E(x_2)$$
$$V(x_1 + x_2) = V(x_1 - x_2) = V(x_1) + V(x_2)$$

Formula for calculating covariance between X and Y

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y)$$

∴ If X, Y are independent $E(XY) = E(X)\,E(Y)$

and hence $\text{Cov}(X, Y) = 0$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.54 In an experiment, positive and negative values are equally likely to occur. The probability of obtaining at most one negative value in five trials is

(a) $\frac{1}{32}$ (b) $\frac{2}{32}$

(c) $\frac{3}{32}$ (d) $\frac{6}{32}$

[CE, GATE-2012, 2 marks]

Solution: (d)

Since negative and positive are equally likely, the distribution of number of negative values is binomial with n = 5 and $p = \frac{1}{2}$

Let X represent number of negative values in 5 trials.

p(at most 1 negative value)

$$= p(x \le 1)$$
$$= p(x = 0) + p(x = 1)$$
$$= 5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 + 5C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = \frac{6}{32}$$

Q.55 Consider the following probability mass function (p.m.f.) of a random variable X.

$$p(X, q) = \begin{cases} q & \text{If } X = 0 \\ 1 - q & \text{If } X = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $q = 0.4$, the variance of X is ________.

[CE, GATE-2015 : 1 Mark, Set-I]

Solution:

Given, $q = 0.4$

X	0	1
$p(X)$	0.4	0.6

Required value = V(X) = $E(X^2) - [E(X)]^2$

$$E(X) = \sum_i X_i p_i = 0\times 0.4 + 1\times 0.6 = 0.6$$

$$E(X^2) = \sum_i X_i^2 p_i = 0^2\times 0.4 + 1^2\times 0.6 = 0.6$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 0.6 - 0.36 = 0.24$$

Q.56 In the following table, x is a discrete random variable and $p(x)$ is the probability density. The standard deviation of x is

x	1	2	3
$p(x)$	0.3	0.6	0.1

(a) 0.18
(b) 0.36
(c) 0.54
(d) 0.6

[ME, 2014 : 2 Marks, Set-2]

Solution : (d)

Mean,

$$\bar{x} = \Sigma\, xp(x) = 1\times 0.3 + 2\times 0.6 + 3\times 0.1 = 1.8$$

Standard deviation,

$$\sigma = \left(\Sigma x^2 p(x) - (\Sigma xp(x))^2\right)^{1/2}$$

$$\therefore \quad \sigma = \left(0.3\times 1^2 + 0.6\times 2^2 + 0.1\times 3^2 - 1.8^2\right)^{1/2}$$

$$= (3.6 - 1.8^2)^{1/2} = (0.36)^{1/2} = 0.6$$

Q.57 A machine produces 0, 1 or 2 defective pieces in a day with associated probability of 1/6, 2/3 and 1/6, respectively. The mean value and the variance of the number of defective pieces produced by the machine in a day, respectively, are

(a) 1 and 1/3
(b) 1/3 and 1
(c) 1 and 4/3
(d) 1/3 and 4/3

[ME, 2014 : 2 Marks, Set-3]

Solution : (a)

x	0	1	2
p(x)	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

$$\text{mean} = \Sigma x\,.\,p(x) = 0\left(\frac{1}{6}\right) + 1\left(\frac{2}{3}\right) + 2\left(\frac{1}{6}\right) = \frac{2}{3} + \frac{2}{6} = \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$E(x^2) = 0\left(\frac{1}{6}\right) + 1\left(\frac{2}{3}\right) + 4\left(\frac{1}{6}\right) = \frac{2}{3} + \frac{4}{6} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$\text{Variance} = E(x^2) - (E(x)^2) = \frac{4}{3} - 1 = \frac{1}{3}$$

Q.58 Two coins R ans S are tossed. The 4 joint events $H_R H_S$, $T_R T_S$, $H_R T_S$, $T_R H_S$ have probabilities 0.28, 0.18, 0.30, 0.24, respectively, where H represents head and T represents tail. Which one of the following is TRUE?

(a) The coin tosses are independent
(b) R is fair, S is not
(c) S is fair, R is not
(d) The coin tosses are dependent

[EE, GATE-2015 : 2 Marks, Set-2]

Solution: (d)

From the given information, we can create a joint probability table as follows:

S \ R	H_R	T_R	
H_S	0.28	0.24	0.52
T_S	0.30	0.18	0.48
	0.58	0.42	1

From the table, we can get

$P(H_R) = 0.58$, $P(T_R) = 0.42$, $P(H_S) = 0.52$

$$P(T_S) = 0.48$$

So, Coins R and S are biased (not fair). So choices (b) and (c) are both false.

The coin tosses are not independent since their probability of heads and tails is not 0.5.

R and S are dependent.

If R and S were independent then all the joint probabilites will be equal to the product of the marginal probabilities.

For example

$$P(H_R \cap H_S) = 0.28$$

$$P(H_R) \cdot P(H_S) = 0.58 \times 0.52 = 0.3016$$

Clearly $P(H_R \cap H_S) \neq P(H_R) \cdot P(H_S)$

So R and S are not independent.

i.e. R and S are dependent. So, choice (a) is false and choice (d) is true.

Q.59 An examination paper has 150 multiple-choice questions of one mark each, with each question having four choices. Each incorrect answer fetches – 0.25 mark. Suppose 1000 students choose all their answers randomly with uniform probablity. The sum total of the expected marks obtained by all these students is

(a) 0
(b) 2550
(c) 7525
(d) 9375

[CS, GATE-2004, 2 marks]

Solution: (d)

Let the marks obtained per question be a random variable X.

Its probability distribution table is given below:

X	1	–0.25
p(X)	1/4	3/4

Expected marks per question $= E(x)$

$= \Sigma X\, p(X)$

$= 1 \times 1/4 + (-0.25) \times 3/4 = 1/4 - 3/16 = 1/16$ marks

Total marks expected for 150 questions $= 1/16 \times 150 = \frac{75}{8}$ marks per student

Total expected marks of 1000 students $= \frac{75}{8} \times 1000 = 9375$ marks

So, correct answer is (d).

Q.60 If the difference between the expectation of the square of a random variable ($E[x^2]$ and the square of the expectation of the random variable $(E[x])^2$ is denoted by R, then

(a) $R = 0$
(b) $R < 0$
(c) $R \geq 0$
(d) $R > 0$

[CS, GATE-2011, 1 marks]

Solution: (c)

$V(x) = E(x^2) - [E(x)]^2 = R$

where V(x) is the variance of x,

Since variance is σ_x^2 and hence never negative, $R \geq 0$.

Q.61 Consider a random variable X that takes values +1 and −1 with probability 0.5 each. The values of the cumulative distribution function F(x) at x = − 1 and +1 are

(a) 0 and 0.5
(b) 0 and 1
(c) 0.5 and 1
(d) 0.25 and 0.75

[CS, GATE-2012, 1 mark]

Solution: (c)

The p.d.f. of the random variable is

x	−1	+1
P(x)	0.5	0.5

The cumulative distribution function F(x) is the probability upto x as given below:

x	−1	+1
F(x)	0.5	1.0

So correct option is (c).

5.3.3.2 Binomial Distribution

Suppose that a trial or an experiment, whose outcome can be classified as either a success or a failure is performed.

Suppose now that n independent trials, each of which results in a success with probability p and in a failure with probability 1 – p, are to be performed.

If X represents the number of successes that occur in the n trials, then X is said to be binomial random variable with parameters (n, p).

The Binomial distribution occurs when experiment performed satisfies the three assumptions of bernoulli trials, which are:

1. Only 2 outcomes are possible, success and failure
2. Probability of success (p) and failure (1 – p) remains same from trial to trial.
3. The trials are statistically independent. i.e The outcome of one trial does not influence subsequent trials. i.e. No memory.

These assumptions are satisfied in following types of problems:

(a) dice problems.

(b) coin toss problems.

(c) sampling with replacement from a finite population.

(d) sampling with or without replacement from an infinite (large) population.

The probability of obtaining x successes from n trials is given by the binomial distribution formula,

$$P(X = x) = nC_x\, p^x\, (1-p)^{n-x}.$$

Where p is the probability of success in any trial and (1-p) = q is the probability of failure.

ILLUSTRATIVE EXAMPLES

Example: 1

10 dice are thrown. What is the probability of getting exactly 2 sixes.

Solution:

$$P(X = 2) = 10C_2\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^8 = 0.2907$$

Example: 2

It is known that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a replacement guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Solution:

If X is the number of defective screws in a packages, then X is a binomial variable with parameters (10, 0.01). Hence, the probability that a package will have to be replaced is:

$$P(X \geq 2) = 1 - [P(X \leq 1)] = 1 - [P\{X = 0\} + P\{X = 1\}]$$

$$= 1 - \left[\binom{10}{0}(0.01)^0(0.99)^{10} + \binom{10}{1}(0.01)^1(0.99)^9\right]$$

$$\approx 0.004$$

Hence only 0.4% of packages will have to be replaced.

For Binomial Distribution:

$$\text{Mean} = E[X] = np$$

$$\text{Variance} = V[X] = np(1-p)$$

Example: 3

100 dice are thrown. How many are expected to fall 6. What is the variance in the number of 6's.

Solution:

$$E(x) = np = 100 \times 1/6 = 16.7 \approx 17$$

So, 17 out of 100 are expected to fall 6.

$$V(x) = np(1-p) = 100 \times 1/6 \times (1 - 1/6) = 13.9$$

So, variance is number of 6's = 13.9.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.62 A box contains 10 screws, 3 of which are defective. Two screws are drawn at random with replacement. The probability that none of the two screws is defective will be

(a) 100%
(b) 50%
(c) 49%
(d) None of these

[CE, GATE-2003, 1 mark]

Solution: (c)

This problem is to be solved by binomial distribution, since although population is finite, sampling is done with replacement and so probability does not change from trial to trial.

Here,
$$n = 2$$
$$x = 0 \text{ (no defective)}$$
$$p = p(\text{defective}) = \frac{3}{10}$$

So,
$$p(x = 0) = 2C_0\left(\frac{3}{10}\right)^0\left(1-\frac{3}{10}\right)^2$$
$$= 0.49 = 49\%$$

Q.63 A lot has 10% defective items. Ten items are chosen randomly from this lot. The probability that exactly 2 of the chosen items are defective is

(a) 0.0036
(b) 0.1937
(c) 0.2234
(d) 0.3874

[ME, GATE-2005, 1 mark]

Solution: (b)

This problem can be done using binomial distribution since population is infinite.

Probability of defective item,
$$p = 0.1$$

Probability of non-defective item,
$$q = 1 - p = 1 - 0.1 = 0.9$$

Probability that exactly 2 of the chosen items are defective
$$= {}^{10}C_2(p)^2(q)^8$$
$$= {}^{10}C_2(0.1)^2(0.9)^8 = 0.1937$$

Q.64 A coin is tossed 4 times. What is the probability of getting heads exactly 3 times?

(a) 1/4
(b) 3/8
(c) 1/2
(d) 3/4

[ME, GATE-2008, 1 mark]

Solution: (a)

Binomial distribution is used, since this problem involves coins.
$$p = p(H) = 0.5$$

Probability of getting head exactly 3 times is
$$p(X = 3) = {}^4C_3(0.5)^3(0.5)^1 = 1/4$$

Q.65 If three coins are tossed simultaneously, the probability of getting at least one head is

(a) 1/8
(b) 3/8
(c) 1/2
(d) 7/8

[ME, GATE-2009, 1 mark]

Solution: (d)

Binomial distribution is used since this problem involves coins.

Here,
$$n = 3$$
$$p = p(H) = 1/2$$
$$x \geq 1$$

Now,
$$p(x \geq 1) = 1 - p(x = 0)$$
$$= 1 - 3C_0\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^3$$
$$= 1 - \frac{1}{8} = \frac{7}{8}$$

Q.66 An unbiased coin is tossed five times. The outcome of each toss is either a head or a tail. The probability of getting at least one head is

(a) $\frac{1}{32}$ (b) $\frac{13}{32}$

(c) $\frac{16}{32}$ (d) $\frac{31}{32}$

[ME, GATE-2011, 2 mark]

Solution: (d)

$$p(x \geq 1) = 1 - p(x = 0)$$
$$= 1 - 5C_0\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}$$

Q.67 Consider an unbiased cubic dice with opposite faces coloured identically and each face coloured red, blue or green such that each colour appears only two times on the dice. If the dice is thrown thrice, the probability of obtaining red colour on top face of the dice at least twice is ____.

[ME, GATE-2014 : 2 Marks, Set-2]

Solution : (0.25 to 0.27)

x	R	B	G
P(x)	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{2}{6}$

$n = 3$ x : red colour

$$p = p(\text{Red}) = \frac{2}{6}$$
$$q = 1 - p = 1 - \frac{2}{6} = \frac{4}{6}$$

Prob. of getting red colour on top face atleast twice is
$$= p(x = 2) + p(x = 3)$$
$$= {}^nC_2 p^2 q^{n-2} + {}^nC_3 p^3 q^{n-3}$$
$$= {}^3C_2\left(\frac{2}{6}\right)^2\left(\frac{4}{6}\right)^1 + {}^3C_3\left(\frac{2}{6}\right)^3\left(\frac{4}{6}\right)^0 = 3\cdot\frac{4}{36}\cdot\frac{4}{6} + 1\cdot\frac{8}{216}$$
$$= \frac{48+8}{216} = \frac{56}{216} = 0.259$$

Q.68 The probability of obtaining at least two "SIX" in throwing a fair dice 4 times is

(a) $\frac{425}{432}$ (b) $\frac{19}{144}$

(c) $\frac{13}{144}$ (d) $\frac{125}{432}$

[ME, GATE-2015 : 2 Marks, Set-3]

Solution: (b)

Let P be the probability that six occurs on a fair dice.

$$\therefore \quad P = \frac{1}{6}$$

$$\therefore \quad q = \frac{5}{6}$$

Let X, be the number of times 'six' occurs,

Probability of obtaining at least two 'six' in throwing a fair dice 4 times is

$$= 1 - \{P(X=0) + P(X=1)\}$$

$$= 1 - \{{}^4C_0\, p^0 q^4 + {}^4C_1\, p^1 q^3\}$$

$$= 1 - \left\{\left(\frac{5}{6}\right)^4 + \left[4 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^3\right]\right\} = 1 - \left\{\frac{125}{144}\right\} = \frac{19}{144}$$

Q.69 A fair coin is tossed independently four times. The probability of the event "the number of times heads show up is more than the number of times tails show up" is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$

(c) $\frac{1}{4}$ (d) $\frac{5}{16}$

[EC, GATE-2010, 2 marks]

Solution: (d)

Coin is tossed 4 times.

p(number of heads > number of tails)

= p(4H & OT or 3H & IT)

= p(Exactly 4 Heads) + p(Exactly 3 Heads)

$$= 4C_4\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^0 + 4C_3\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^1 = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

Q.70 If a fair coin is tossed four times. What is the probability that two heads and two tails will result?

(a) 3/8 (b) 1/2

(c) 5/8 (d) 3/4

[CS, GATE-2004, 1 mark]

Solution: (a)

The condition getting 2 heads and 2 tails is same as getting exactly 2 heads out of 4 tosses.

Given, $p = P(H) = 1/2$

Applying the formula for binomial distribution, we get,

$$P(X=2) = {}^4C_2(1/2)^2\left(1-\frac{1}{2}\right)^{4-2} = {}^4C_2\left(\frac{1}{2}\right)^2(1/2)^2 = \frac{4C_2}{2^4} = \frac{6}{16} = \frac{3}{8}$$

Q.71 Two n bit binary strings, S_1 and S_2 are chosen randomly with uniform probability. The probability that the Hamming distance between these strings (the number of bit positions where the two strings differ) is equal to d is

(a) ${}^nC_d/2^n$
(b) ${}^nC_d/2^d$
(c) $d/2^n$
(d) $1/2^d$

[CS, GATE-2004, 2 marks]

Solution: (a)

If hamming distance between two n bit strings is d, we are asking that d out of n trials to be success (success here means that the bits are different). So this is a binomial distribution with n trials and d successes and probability of success

$$p = 2/4 = 1/2$$

(Since out of the 4 possibilities [(0, 0), (0, 1), (1, 0), (1, 1)] only two of then (0, 1) and (1, 0) are success)

So, $$p(X = d) = n\,C_d\,(1/2)^d\,(1/2)^{n-d} = \frac{n\,C_d}{2^n}$$

Correct choice is therefore (a).

Q.72 For each element in a set of size 2n, an unbiased coin is tossed. All the 2n coin tossed are independent. An element is chosen if the corresponding coin toss were head. The probability that exactly n elements are chosen is

(a) $\binom{2n}{n}\Big/4^n$
(b) $\binom{2n}{n}\Big/2^n$
(c) $1\Big/\binom{2n}{n}$
(d) $\frac{1}{2}$

[CS, GATE-2006, 2 marks]

Solution: (a)

The probability that exactly n elements are chosen

$$= \text{The probability of getting n heads out of 2n tosses}$$
$$= 2n\,C_n\,(1/2)^n\,(1/2)^{2n-n} \quad \text{(Binomial formula)}$$
$$= 2n\,C_n\,(1/2)^n\,(1/2)^n$$
$$= \frac{{}^{2n}C_n}{2^{2n}} = \frac{{}^{2n}C_n}{(2^2)^n} = \frac{{}^{2n}C_n}{4^n}$$

Q.73 The probability that a thermistor randomly picked up from a production unit is defective is 0.1. The probability that out of 10 thermistors randomly picked up, 3 are defective is

(a) 0.001
(b) 0.057
(c) 0.107
(d) 0.3

[IN, GATE-2015 : 2 Marks]

Solution: (b)

$$\text{Probability} = {}^{10}C_3\,(0.1)^3\,(0.9)^7 = 0.057$$

Q.74 The probability that a screw manufactured by a company is defective is 0.1. The company sells screws in packets containing 5 screws and gives a guarantee of replacement if one or more screws in the packet are found to be defective The probability that a packet would have to be replaced is____.

[ME, 2016 : 2 Marks, Set-2]

Solution:

$n = 5$, $P = 0.1$, $q = 0.9$

X : no of defectives

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 5,(0.1)^{\circ}(0.9)^5 = 1 - (0^{\circ}.9)^5 = 0.4095$$

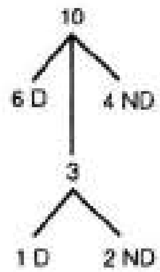
5.3.3.3 Hypergeometric Distribution

If the probability changes from trial to trial, one of the assumptions of binomial distribution gets violated and hence binomial distribution cannot be used. In such lases hypergeometric distribution is used. This is particularly used is cases of sampling without replacement from a finite population.

ILLUSTRATIVE EXAMPLES

Example:

There are 10 markers on a table, of which 6 are defective and 4 are not deflective. If 3 are randomly taken from above lot, what is the probability that exactly 1 of markers is defective ?



Solution:

The above problem is tackled by hypergeometric distribution as follows.
D is defective and ND is non defective.

$$p(X = 1) = \frac{6C_1 \times 4C_2}{10C_3} = 0.3$$

The above problem can be generalised into a distribution if we make X as the number of defective markers.

X can now take the values 0, 1, 2 or 3.

$$p(X = x) = \frac{6C_x \times 4C_{3-x}}{10C_3}$$

This is the hypergeometric distribution for above problem.

from above formula, we can calculate the following:

$$p(x = 1) = \frac{6C_1 \times 4C_2}{10C_3}$$

$$p(x \geq 1) = p(x = 0) + p(x = 1) = \frac{6C_0 \times 4C_3}{10C_3} + \frac{6C_1 \times 4C_2}{10C_3}$$

$$p(x \geq 1) = 1 - p(x = 0) = 1 - \left[\frac{6C_0 \times 4C_3}{10C_3}\right]$$

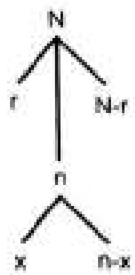
The hypergeometric distribution can be written is a more general way as follows.

Consider N objects of which r are of type 1 and N-r are of type 2.

from this n objects are drawn without replacement. What is the probability that x objects drawn are of type 1?

The diagram for above problem is

$$p(X = x) = \frac{rC_x \times {}^{N-r}C_{n-x}}{NC_n}$$



This is the general formula for hypergeometric distribution.

The expected value of this distribution is given by.

$$E(x) = n^* \left(\frac{r}{N}\right)$$

ILLUSTRATIVE EXAMPLES FROM GATE

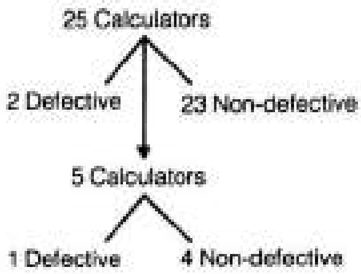
Q.75 There are 25 calculators in a box. Two of them are defective. Suppose 5 calculators are randomly picked for inspection (i.e., each has the same chance of being selected), what is the probability that only one of the defective calculators will be included in the inspection?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{5}$ **[CE, GATE-2006, 2 marks]**

Solution: (b)

Since population is finite, hypergeometric distribution is applicable



$$p(\text{1 defective in 5 calculators}) = \frac{2C_1 \times 23C_4}{25C_5} = \frac{1}{3}$$

Q.76 A box contains 5 black and 5 red balls. Two balls are randomly picked one after another from the box, without replacement. The probability for both balls being red is

(a) $\frac{1}{90}$ (b) $\frac{1}{2}$

(c) $\frac{19}{90}$ (d) $\frac{2}{9}$

[ME, GATE-2003, 2 marks]

Solution: (d)

Probability of drawing two red balls

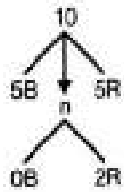
$$= p(\text{first is red}) \times p(\text{second is red given that first is red})$$

$$= \frac{5}{10} \times \frac{4}{9} = \frac{2}{9}$$

Alternatively this problem can be done as hypergeometric distribution, since it is sampling without replacement from finite population.

$$\text{From above diagram, } p(X = 2) = \frac{5C \times 5C_2}{10C_2}$$

$$= \frac{5 \times 4}{10 \times 9} = \frac{2}{9}$$



Q.77 From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be Kings, if first card in NOT replaced?

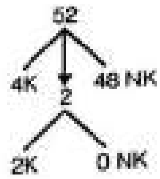
(a) $\frac{1}{26}$ (b) $\frac{1}{52}$

(c) $\frac{1}{169}$ (d) $\frac{1}{221}$

[ME, GATE-2004, 2 marks]

Solution: (d)

Problems can be solved by hypergeometric distribution as follows:



$$p(X=2) = \frac{4C_2 \times 48C_0}{52C_2} = \frac{1}{221}$$

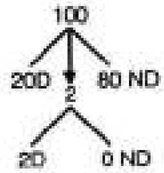
Q.78 A box contains 20 defective items and 80 non-defective items. If two items are selected at random without replacement, what will be the probability that both items are defective?

(a) $\frac{1}{5}$ (b) $\frac{1}{25}$

(c) $\frac{20}{99}$ (d) $\frac{19}{495}$ **[ME, GATE-2006, 1 mark]**

Solution: (d)

Problem can be solved by hypergeometric distribution



$$p(X=2) = \frac{20C_2 \times 80C_0}{100C_2} = \frac{19}{495}$$

Q.79 A box contains 4 red balls and 6 black balls. Three balls are selected randomly from the box one after another, without replacement. The probability that the selected set contains one red ball and two black balls is

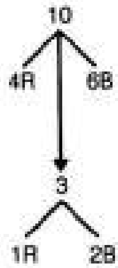
(a) 1/20 (b) 1/12

(c) 3/10 (d) 1/2

[ME, GATE-2012, 2 marks]

Solution: (d)

The problem can be represented by the following diagram.



$$p(1R \text{ and } 2B) = \frac{4C_1 \times 6C_2}{10C_3} = \frac{60}{120} = \frac{1}{2}$$

Q.80 The security system at an IT office is composed of 10 computers of which exactly four are working. To check whether the system is functional, the officials inspect four of the computers picked at random (without replacement). The system is deemed functional if at least three of the four computers inspected are working. Let the probability that the system is deemed functional be denoted by p. Then 100p = ________.

[CS, GATE-2014 : 1 Mark, Set-2]

Solution: (11.90)

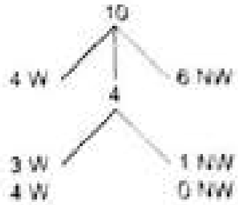
The tree diagram for the problem is shown below:

$$\text{Required probability} = \frac{{}^4C_3 \cdot {}^6C_1}{{}^{10}C_4} + \frac{{}^4C_4 \cdot {}^6C_0}{{}^{10}C_4}$$

$$= \frac{24}{210} + \frac{1}{210} = \frac{25}{210}$$

$$p = 0.1190$$

$$\Rightarrow \quad 100\,p = 11.90$$



5.3.3.4 Geometric Distribution

Consider repeated trials of a Bernoulli experiment ϵ with probability P of success and q = 1 − P of failure. Let x denote the number of times ϵ must be repeated until finally obtaining a success. The distribution of random variable x is given as follows:

k	1	2	3	4	5	....
P(k)	P	qP	q^2P	q^3P	q^4P	

The experiment ϵ will be repeated k times only in the case that there is a sequence of k − 1 failures followed by a success.

$$P(k) = P(x = k) = q^{k-1}P$$

The geometric distribution is characterized by a single parameter P.

Points to Remember:

Let x be a geometric random variable with distribution GEO(P). Then

1. $E(x) = \frac{1}{P}$
2. $Var(x) = \frac{q}{P^2}$
3. Cumulative distribution $F(k) = -1 - q^k$
4. $P(x > r) = q^r$

Geometric distribution possesses "no-memory" or "lack of memory" property which can be stated as

$$P(x > a + r \mid x > a) = P(x > r)$$

1. Suppose the probability that team A wins each game in a tournament is 60 percent. A plays until it loses.
   (a) Find the expected number E of games that A plays
   (b) Find the probability P that A plays in at least 4 games
   (c) Find the probability P that A wins the tournament if the tournament has 64 teams. (Thus, a team winning 6 times wins the tournament).

Sol. 1

This is a geometric distribution with P = 0.4 and q = 0.6 (A plays until A loses)

(a) Since $E(x) = \frac{1}{P} = \frac{1}{0.4} = 2.5$

(b) The only way A plays at least 4 games is if A wins the first 3 games. Thus.
$P = P(x > 3) = q^3 = (0.6)^3 = 0.216 = 21.6\%$

(c) Here A must win all 6 games:

So
$$P = (0.6)^6 = 0.0467 = 4.67\%$$

5.3.3.5 Poisson Distribution

A random variable X, taking on one of the values 0, 1, 2 .... is said to be a Poisson random variable with parameter λ if for some $\lambda > 0$,

$$P(x = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

For Poisson distribution:

$$\text{Mean} = E(x) = \lambda$$
$$\text{Variance} = V(x) = \lambda$$

Therefore, expected value and variance of a Poisson random variable are both equal to its parameter λ. Here λ is average number of occurrences of event in an observation period Δt. So, $\lambda = \alpha\,\Delta t$ where α is no of occurrences of event per unit time.

ILLUSTRATIVE EXAMPLES

Example: 1

A certain airport receives on an average of 4 aircrafts per hour. What is the probability that no aircraft lands in a particular 2 hr period.

Solution:

Given equation, α = rate of occurrence of event per unit time = 4/hr

λ = avg. no of occurrences of event in specified observation period $= \alpha\Delta t$

In this case α = 4/hr and Δt = 2h

∴ So, $\lambda = 4 \times 2 = 8$

Now we wish that no aircraft should land for 2 hrs. i.e. $x = 0$

$$P(x = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-8}8^0}{0!} = e^{-8}$$

Frequently, Poisson distribution is used to approximate binomial distribution when n is very large & p is very small. Notice that direct computation of $nC_x p^x (1-p)^{n-x}$ may be erroneous or impossible when n is very large & p is very small. Hence, we resort to a Poisson approximation with $\lambda = np$.

Example: 2

A certain company sells tractors which fail at a rate of 1 out of 1000. If 500 tractors are purchased from this company what is the probability of 2 of them failing within first year.

Solution:

$$\lambda = np = 500 \times \frac{1}{1000} = \frac{1}{2}$$

$$P(x = 2) = \frac{e^{-1/2}(1/2)^2}{2!} = 0.07582$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.81 Suppose p is the number of cars per minute passing through a certain road junction between 5 PM, and p has Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars during any given minute in this interval?

(a) $8/(2e^3)$ (b) $9/(2e^3)$

(c) $17/(2e^3)$ (d) $26/(2e^3)$ **[CS, GATE-2013, 1 Mark]**

Solution: (c)

Poisson formula for $(P = x)$ given as

$$\frac{e^{-\lambda}\lambda^x}{x!}$$

λ : mean of Poisson distribution = 3 (given)

Probability of observing fewer than 3 cars.

$(P = 0) + (P = 1) + (P = 2)$

$$\frac{e^{-3}\lambda^0}{0!}+\frac{e^{-3}\lambda^1}{1!}+\frac{e^{-3}\lambda^2}{2!} = \frac{17}{2e^3}$$

(c) is correct option.

Q.82 A traffic office imposes on an average 5 number of penalties daily on traffic violators. Assume that the number of penalties on different days is independent and follows a Poisson distribution. The probability that there will be less than 4 penalties in a day is __________.

[CE, GATE-2014 : 2 Marks, Set-1]

Solution :

Mean $\lambda = 5$

$$P(x < 4) = p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3)$$

$$= \frac{e^{-5}5^0}{0!}+\frac{e^{-5}5^1}{1!}+\frac{e^{-5}5^2}{1!}+\frac{e^{-5}5^3}{3!}$$

$$= e^{-5}\left[1+5+\frac{25}{2}+\frac{125}{6}\right] = e^{-5}\left(\frac{118}{3}\right) = 0.265$$

Q.83 The number of accidents occurring in a plant in a month follows Poisson distribution with mean as 5.2. The probability of occurrence of less than 2 accidents in the plant during a randomly selected month is

(a) 0.029 (b) 0.034

(c) 0.039 (d) 0.044

[ME, GATE-2014 : 2 Marks, Set-2]

Solution : (b)

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

As λ(mean) = 5.2

$$P(x < 2) = P(0) + P(1) = e^{-5.2}\left[\frac{5.2^0}{0!}+\frac{5.2^1}{1!}\right]$$

$$\therefore \quad P(x < 2) = \frac{6.2}{e^{5.2}} = 0.0342$$

Q.84 The second moment of a Poisson-distributed random variable is 2. The mean of the random variable is ______.

[EC, 2016 : 1 Mark, Set-1]

Solution:

In Poisson distribution,

$$\text{Mean} = \text{First moment} = \lambda$$

$$\text{second moment} = \lambda^2 + \lambda$$

Given that second moment is 2

$$\therefore \quad \lambda^2 + \lambda = 2$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda + 2)(\lambda - 1) = 0$$

$$\lambda = 1$$

Q.85 Consider a Poisson distribution for the tossing of a biased coin. The mean for this distribution is μ. The standard deviation for this distribution is given by

(a) $\sqrt{\mu}$ (b) μ^2

(c) μ (d) $\frac{1}{\mu}$

[ME, 2016 : 1 Marks, Set-1]

Solution: (a)

In poisson distribution mean = Variance

Given that mean = Variance = m

$$\text{Standard deviation} = \sqrt{\text{Variance}} = \sqrt{\mu}$$

Continuous Distributions:

1. General Continuous Distribution
2. Uniform Distribution
3. Exponential Distribution
4. Normal Distribution
5. Standard Normal Distribution

5.3.3.6 General Continuous Distribution

Let X be a continuous random variable. A continuous distribution of X can be defined by a probability density function f(x) which is such a function such that

$$p(-\infty \le X \le \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$

The expected value of x is given by

$$\mu_x = E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

i.e.

$$V(x) = E(x^2) - [E(x)]^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} x\, f(x)dx\right]^2$$

$$\sigma_x^2 = \sqrt{V(x)}$$

The cumulative probability function (sometimes also called as probability distribution function), is given by F(x), where

$$F(x) = p(X \le x) = \int_{-\infty}^{x} f(x)dx$$

Note: From distribution function we can get probability density franction by formula below:

$$f(x) = \frac{dF}{dx}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.86 If probability density function of a random variable X is

$$f(x) = x^2 \text{ for } -1 \le x \le 1, \text{ and}$$
$$= 0 \text{ for any other value of } x$$

then, the percentage probability $P\left(-\frac{1}{3} \le x \le \frac{1}{3}\right)$ is

(a) 0.247
(b) 2.47
(c) 24.7
(d) 247

[CE, GATE-2008, 2 marks]

Solution: (b)

Given,

$$f(x) = x^2 \quad -1 \le x \le 1$$
$$= 0 \quad \text{elsewhere}$$

$$p\left(-\frac{1}{3} \le x \le \frac{1}{3}\right) = \int_{-\frac{1}{3}}^{\frac{1}{3}} f(x)\,dx = \int_{-\frac{1}{3}}^{\frac{1}{3}} x^2\,dx = \left[\frac{x^3}{3}\right]_{-\frac{1}{3}}^{\frac{1}{3}} = \frac{2}{81}$$

The probability expressed in percentage, $p = \frac{2}{81} \times 100 = 2.469\% = 2.47\%$

Q.87 Find the value of λ such that function f(x) is valid probability density function

$f(x) = \lambda\ (x-1)(2-x)$ for $1 \le x \le 2$

$= 0$ otherwise

[CE, GATE-2013, 2 Mark]

Solution:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$f(x) = \begin{cases} \lambda\left(-x^2+3x-2\right) & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad \int_1^2 \lambda\left(-x^2+3x-2\right)dx = 1$$

$$\Rightarrow \quad \lambda\left[-\frac{x^3}{3}+3\frac{x^2}{2}-2x\right]_1^2 = 1$$

$$\Rightarrow \lambda\left[-\left(\frac{8}{3}-\frac{1}{3}\right)+\frac{3}{2}(4-1)-2(2-1)\right] = 1$$

$$\Rightarrow \quad \lambda\left[-\frac{7}{3}+\frac{9}{2}-2\right] = 1$$

$$\Rightarrow \quad \lambda\left[\frac{-14+27-12}{6}\right] = 1$$

$$\Rightarrow \quad \lambda = \frac{6}{1} = 6$$

$$\lambda = 6$$

Q.88 The probability density function of evaporation E on any day during a year in a watershed is given by

$$f(E) = \begin{cases} \frac{1}{5} & 0 \le E \le 5 \text{ mm/day} \\ 0 & \text{otherwise} \end{cases}$$

The probability that E lies in between 2 and 4 mm/day in a day in the watershed is (in decimal) ______.

[CE, GATE-2014 : 1 Mark, Set-1]

Solution :

$$f(E) = \begin{cases} \frac{1}{5} & 0 \le E \le \text{mm/day} \\ 0 & \text{Otherwise} \end{cases}$$

$$P(2 < E < 4) = \int_2^4 f(E)dE = \int_2^4 \frac{1}{5}dE = \frac{1}{5}[E]_2^4 = \frac{1}{5}(4-2) = \frac{2}{5} = 0.4$$

Q.89 The probability density function of a random variable, x is

$$f(x) = \frac{x}{4}(4-x^2) \text{ for } 0 \le x \le 2$$
$$= 0$$

The mean, μ_x of the random variable is ______.

[CE, GATE-2015 : 2 Marks, Set-II]

Solution :

$$f(x) = \begin{cases} \frac{x}{4}(4-x^2), & 0 \le x \le 2 \\ 0 & , \text{Otherwise} \end{cases}$$

$$\mu_x = \int_0^2 xf(x)dx$$

$$\therefore \text{Mean}(\mu_x) = \int_0^2 x\frac{x}{4}(4-x^2)dx = \int_0^2\left(x^2-\frac{x^4}{4}\right)dx$$

$$= \left[\frac{x^3}{3}-\frac{x^5}{20}\right]_0^2 = \frac{8}{3}-\frac{32}{20} = \frac{16}{15} = 1.066$$

Q.90 Consider the continuous random variable with probability density function

$$f(t) = 1 + t \text{ for } -1 \le t \le 0$$
$$= 1 - t \text{ for } 0 \le t \le 1$$

The standard deviation of the random variable is

(a) $\frac{1}{\sqrt{3}}$
(b) $\frac{1}{\sqrt{6}}$
(c) $\frac{1}{3}$
(d) $\frac{1}{6}$

[ME, GATE-2006, 2 marks]

Solution: (b)

Mean $\mu_1 = E(t) = \int_{-\infty}^{\infty} t \cdot f(t) \cdot dt = \int_{-1}^{1} t \cdot f(t) \cdot dt = \int_{-1}^{0} t(1+t)dt + \int_{0}^{1} t(1-t)dt$

$$= \left[\frac{t^2}{2} + \frac{t^3}{3}\right]_{-1}^{0} + \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_{0}^{1} = -\left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= -\left[\frac{1}{6} + \frac{1}{6}\right] = 0$$

$$\text{Variance} = E(t^2) - [E(t)]^2$$

$$= \int_{-\infty}^{\infty} t^2 \, f(t)dt - [E(t)]^2 = \int_{-\infty}^{\infty} t^2 \, f(t)dt - (0)^2$$

$$= \int_{-\infty}^{\infty} t^2 \, f(t)dt = \int_{-1}^{0} t^2(1+t)dt + \int_{0}^{1} t^2(1-t)dt$$

$$= \int_{-1}^{0} (t^2 + t^3) \cdot dt + \int_{0}^{1} t^2 (1-t)dt = \left[\frac{t^3}{3} + \frac{t^4}{4}\right]_{-1}^{0} + \left[\frac{t^3}{3} - \frac{t^4}{4}\right]_{0}^{1}$$

$$= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$\text{Standard deviation} = \sqrt{\text{variance}} = \frac{1}{\sqrt{6}}$$

Q.91 Let X and Y be two independent random variables. Which one of the relations between expectation (E), variance (Var) and covariance (Cov) given below is FALSE?

(a) $E(XY) = E(X)\,E(Y)$
(b) $Cov(X, Y) = 0$
(c) $Var(X + Y) = Var(X) + Var(Y)$
(d) $E(X^2 Y^2) = (E(X))^2 (E(Y))^2$

[ME, GATE-2007, 2 marks]

Solution: (d)

(a) is true, (b) is true, (c) is true.

(d) is false.

since, $E(X^2 Y^2) = E(X^2)\,E(Y^2)$

But since X is not independent of X,

$$E(X^2) \ne [E(X)]^2$$

$\therefore$
$$E(X^2 Y^2) = E(X^2)\,E(Y^2)$$
$$\ne [E(X)]^2 [E(Y)]^2$$

Q.92 A continuous random variable X has a probability density function $f(x) = e^{-x}$, $0 < x < \infty$. Then P{X > 1} is

(a) 0.368 (b) 0.5

(c) 0.632 (d) 1.0

[EE, GATE-2013, 1 Mark]

Solution: (a)

$$P = \int_1^\infty f(x)dx = \int_1^\infty e^{-x}dx = -e^{-x}\Big|_1^\infty = -(e^{-\infty} - e^{-1}) = e^{-1} = 0.368$$

Q.93 Let X be a random variable with probability density function

$$f(x) = \begin{cases} 0.2, & \text{for } |x| \le 1 \\ 0.1, & \text{for } 1 < |x| \le 4 \\ 0, & \text{otherwise} \end{cases}$$

The probability P(0.5 < X < 5) is ______.

[EE, GATE-2014 : 2 Marks, Set-2]

Solution :

$$\text{Probability } (0.5 < n < 5) = \int_{0.5}^{5} f(x)dx$$

$$= \int_{0.5}^{1} 0.2dx + \int_1^4 0.1\,dx + \int_4^5 0\,dx$$

$$= 0.2[1 - 0.5] + 0.1[4 - 1] + 0[5 - 4]$$

$$= 0.2 \times 0.5 + 0.1 \times 3 = 0.1 + 0.3 = 0.4$$

Q.94 A random variable X has probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If the expected value $E[X] = 2/3$, then $Pr[X < 0.5]$ is ______.

[EE, GATE-2015 : 1 Mark, Set-1]

Solution: (0.25)

$$f(x) = \begin{cases} a + bx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Now given $E(X) = 2/3$

$$\Rightarrow \quad \int_0^1 xf(x)dx = \frac{2}{3}$$

$$\Rightarrow \quad = \int_0^1 x(a + bx)\,dx = \frac{2}{3}$$

$$\Rightarrow \quad a\left(\frac{x^2}{2}\right)_0^1 + b\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3}$$

$$a\left(\frac{1}{2}\right)+b\left(\frac{1}{3}\right) = \frac{2}{3}$$

$\Rightarrow$ $$3a + 2b = 4 \quad \text{...(i)}$$

Now, $$\int_0^1 f(x)dx = 1$$ (Total probability is always equal to 1)

$\Rightarrow$ $$= \int_0^1 (a+bx)\,dx = \left(ax + \frac{bx^2}{2}\right)_0^1 = 1$$

$\Rightarrow$ $$a + \frac{b}{2} = 1$$

$\Rightarrow$ $$2a + b = 2 \quad \text{...(ii)}$$

Now solving (i) and (ii), we get

$$a = 0,\ b = 2$$

So $f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Now we need $$\int_0^{1/2} 2x\,dx = \frac{1}{4}$$

Q.95 A continuous random variable X has a probability density $f(x) = e^{-x}$, $0 < x < \infty$. Then $P(X > 1)$ is

(a) 0.368
(b) 0.5
(c) 0.632
(d) 1.0

[IN, GATE-2013 : 1 mark]

Solution: (a)

$$P = \int_1^{\infty} f(x)dx = \int_1^{\infty} e^{-x}dx = e^{-x}\Big|_{\infty}^{1} = e^{-1} = 0.368$$

Q.96 A probability density function is of the form $p(x) = Ke^{-\alpha|x|}$, $x \in (-\infty, \infty)$.
The value of K is

(a) 0.5
(b) 1
(c) 0.5 α
(d) α

[EC, GATE-2006, 1 mark]

Solution: (c)

$$\int_{-\infty}^{\infty} p(x).dx = 1$$

$$\int_{-\infty}^{\infty} K.e^{-\alpha|x|}.dx = 1$$

$$\int_{-\infty}^{0} K.e^{\alpha x}dx + \int_0^{\infty} K.e^{-\alpha x} = 1$$

$\Rightarrow$ $$\frac{K}{\alpha}\left(e^{\alpha x}\right)_{-\infty}^{0} + \frac{K}{-\alpha}\left(e^{-\alpha x}\right)_0^{\infty} = 1$$

$\Rightarrow$ $$\frac{K}{\alpha} + \frac{K}{\alpha} = 1$$

$$2K = \alpha$$

$\Rightarrow$ $$K = 0.5\alpha$$

Q.97 A program consists of two modules executed sequentially. Let $f_1(t)$ and $f_2(t)$ respectively denote the probability density functions of time taken to execute the two modules. The probability density function of the overall time taken to execute the program is given by

(a) $f_1(t) + f_2(t)$

(b) $\int_0^t f_1(x) f_2(x) dx$

(c) $\int_0^t f_1(x) f_2(t-x) dx$

(d) $\max\{f_1(t), f_2(t)\}$

[CS, GATE-2003, 2 marks]

Solution: (c)

Let the time taken for first and second modules be represented by x and y and total time = t.

$\therefore$ t = x + y is a random variable.

Now the joint density function,

$$g(t) = \int_0^t f(x, y)dx = \int_0^t f(x, t-x)dx = \int_0^t f_1(x)\, f_2(t-x)dx$$

which is also called as convolution of f_1 and f_2, abbreviated as $f_1 * f_2$.

Correct answer is therefore, choice (c).

Q.98 Let f(x) be the continuous probability density function of a random variable X. The probability that $a < X \le b$, is

(a) $f(b - a)$

(b) $f(b) - f(a)$

(c) $\int_a^b f(x)dx$

(d) $\int_a^b x f(x)dx$

[CS, GATE-2005, 1 mark]

Solution: (c)

If f(x) is the continuous probability density function of a random variable X then,

$$p(a < x \le b) = p(a \le x \le b) = \int_a^b f(x)dx$$

Q.99 A probability density function on the interval [*a*, 1] is given by $1/x^2$ and outside this interval the value of the function is zero. The value of *a* is_____.

[CS, 2016 : 1 Mark, Set-1]

Solution:

Given,
$$f(x) = \frac{1}{x^2} \quad a \le x \le 1$$
$$= 0 \quad \text{elsewhere}$$

So
$$\int_a^1 f(x) = 1$$

$\Rightarrow$
$$\int_a^1 \frac{1}{x^2} = 1$$

$\Rightarrow$
$$\left[\frac{-1}{x}\right]_a^1 = 1$$

$$-\left[\frac{1}{1}-\frac{1}{a}\right] = 1$$

$$\Rightarrow \quad \frac{1}{a} = 2$$

$$\Rightarrow \quad a = \frac{1}{2} = 0.5$$

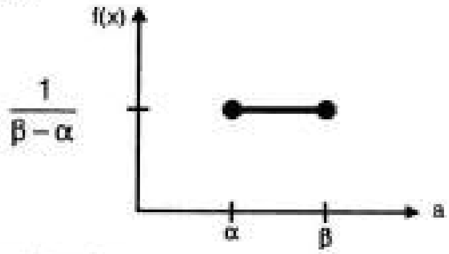
5.3.3.7 Uniform Distribution

In general we say that X is a uniform random variable on the interval (a, b) if its probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Since f(x) is a constant, all values of x between α and β are equally likely (uniform).

Graphical Representation:



For Discrete Uniform Distribution:

$$\text{Mean} = E[X] = \frac{\beta+\alpha}{2}$$

$$\text{Variance} = V(X) = \frac{(\beta-\alpha)^2}{12}$$

ILLUSTRATIVE EXAMPLES

Example:

If X is uniformly distributed over (0, 10), calculate the probability that

(a) $X < 3$

(b) $X > 6$

(c) $3 < X < 8$.

Solution:

$$f(x) = \frac{1}{10-0} = \frac{1}{10}$$

$$P[X < 3] = \int_0^3 \frac{1}{10}dx = \frac{3}{10}$$

$$P[X > 6] = \int_6^{10} \frac{1}{10}dx = \frac{4}{10}$$

$$P[3 < X < 8] = \int_3^8 \frac{1}{10}dx = \frac{1}{2}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.100 The standard deviation of a uniformly distributed random variable between 0 and 1 is

(a) $\frac{1}{\sqrt{12}}$
(b) $\frac{1}{\sqrt{3}}$
(c) $\frac{5}{\sqrt{12}}$
(d) $\frac{7}{\sqrt{12}}$ **[ME, GATE-2009, 2 marks]**

Solution: (a)

$$\sigma = \sqrt{\frac{(\beta-\alpha)^2}{12}} = \sqrt{\frac{(1-0)^2}{12}} = \frac{1}{\sqrt{12}}$$

Q.101 If P and Q are two random events, then the following is TRUE

(a) Independence of P and Q implies that probability $(P \cap Q) = 0$
(b) Probability $(P \cup Q) \geq$ Probability (P) + Probability (Q)
(c) If P and Q are mutually exclusive, then they must be independent
(d) Probability $(P \cap Q) \leq$ Probability (P) **[EE, GATE-2005, 1 mark]**

Solution: (d)

(a) is false since if P & Q are independent

$$pr(P \cap Q) = pr(P) * pr(Q)$$

which need not be zero.

(b) is false since $pr(P \cup Q) = pr(P) + pr(Q) - pr(P \cap Q)$

$\therefore$ $pr(P \cup Q) \leq pr(P) + pr(Q)$

(c) is false since independence and mutually exclusive are unrelated properties.

(d) is true

since $P \cap Q \subseteq P$

$\Rightarrow$ $n(P \cap Q) \leq n(P)$

$\div$ both sides by n(S) we get,

$$\frac{n(P \cap Q)}{n(S)} \leq \frac{n(P)}{n(S)}$$

$\Rightarrow$ $pr(P \cap Q) \leq pr(P)$

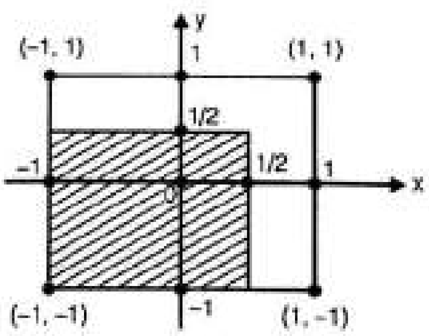
Q.102 Two independent random variables X and Y are uniformly distributed in the interval [–1, 1]. The probability that max [X, Y] is less than 1/2 is

(a) 3/4
(b) 9/16
(c) 1/4
(d) 2/3 **[EE, GATE-2012, 1 mark]**

Solution: (b)

$-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ is the entire rectangle.

The region in which maximum of {x, y} is less than $\frac{1}{2}$ is shown below as shaded region inside this rectangle.



$$p\left(\max\{x, y\} < \frac{1}{2}\right) = \frac{\text{Area of shaded region}}{\text{Area of entire rectangle}}$$

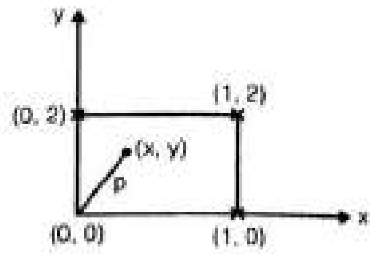
$$= \frac{\frac{3}{2} \times \frac{3}{2}}{2 \times 2} = \frac{9}{16}$$

Q.103 A point is randomly selected with uniform probability in the X-Y plane within the rectangle with corners at (0, 0), (1, 0), (1, 2) and (0, 2). If p is the length of the position vector of the point, the expected value of p^2 is

(a) 2/3 (b) 1

(c) 4/3 (d) 5/3 **[CS, GATE-2004, 2 marks]**

Solution: (d)



Length of position vector of point = p = $\sqrt{x^2 + y^2}$

$$p^2 = x^2 + y^2$$

$$E(p^2) = E(x^2 + y^2) = E(x^2) + E(y^2)$$

Now x and y are uniformly distributed $0 \le x \le 1$ and $0 \le y \le 2$

Probability density function of x = $\frac{1}{1-0} = 1$

Probability density function of y = $\frac{1}{2-0} = 1/2$

$$E(x^2) = \int_0^1 x^2 \, p(x)\, dx = \int_0^1 x^2 \cdot 1 \cdot dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E(y^2) = \int_0^2 y^2 \, p(y)\, dy = \int_0^2 y^2 \cdot 1/2 \cdot dy = \left[\frac{y^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$$

$$\therefore \quad E(p^2) = E(x^2) + E(y^2) = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$$

Q.104 Two random variables X and Y are distributed according to

$$f_{X,Y}(x,y) = \begin{cases} (x+y) & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The probability $P(X + Y \le 1)$ is ______. **[EC, 2016 : 2 Marks, Set-2]**

Solution:

$$P(X + Y \le 1) = \int_{x=0}^{1} \int_{y=0}^{(1-x)} f_{xy}(x, y)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (x+y)\, dx\, dy = \int_{x=0}^{1} \left(xy + \frac{y^2}{2}\right)_0^{1-x}$$

$$= \int_{x=0}^{1} \left(x(1-x) + \frac{(1-x)^2}{2}\right) dx = \int_{x=0}^{1} \left(\frac{1}{2} - \frac{x^2}{2}\right) dx = \left(\frac{x}{2} - \frac{x^3}{6}\right)_0^1$$

$$= \frac{1}{2} - \frac{1}{6} = \frac{1}{3} = 0.33$$

Q.105 Probability density function of a random variable X is given below

$$f(x) = \begin{bmatrix} 0.25 & \text{if } 1 \le x \ge 5 \\ 0 & \text{otherwise} \end{bmatrix}$$

$P(X \le 4)$

(a) $\frac{3}{4}$
(b) $\frac{1}{2}$
(c) $\frac{1}{4}$
(d) $\frac{1}{8}$

[CE, 2016 : 2 Marks, Set-I]

Solution: (a)

$$P(x \le 4) = \int_{-\infty}^{4} f(x)dx = \int_{-\infty}^{1}(0)dx + \int_{1}^{4}(0.25)dx + \int_{4}^{\infty}(0)dx = \frac{1}{4}(x)_1^4 = \frac{1}{4}(4-1) = \frac{3}{4}$$

5.3.3.7 Exponential Distribution

A continuous random variable whose probability density function is given for some $\lambda > 0$ by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is said to be exponential random variable with parameter λ. The cumulative distributive function F(a) of an exponential random variable is given by:

$$F(a) = P(X \le a) = \int_0^a \lambda e^{-\lambda x}dx = \left(-e^{-\lambda x}\right)_0^a = 1 - e^{-\lambda a}, \; a \ge 0$$

For Exponential Distribution:

$$\text{Mean} = E[X] = 1/\lambda$$
$$\text{Variance} = v(x) = 1/\lambda^2$$

ILLUSTRATIVE EXAMPLES

Example:

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait,

(a) More than 10 minutes

(b) Between 10 and 20 minutes.

Solution:

Letting X denote the length of the call made by the person in the booth, we have that the desired probabilities are:

(a)
$$P\{X > 10\} = 1 - P(x < 10)$$
$$= 1 - F(10) = 1 - (1 - e^{-\lambda \times 10})$$
$$= e^{-10}\lambda = e^{-1} = 0.368$$

(b)
$$P\{10 < X < 20\} = F(20) - F(10)$$
$$= (1 - e^{-20\lambda}) - (1 - e^{-10\lambda}) = e^{-1} - e^{-2} = 0.233$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.106 Let the probability density function of a random variable, X, be given as:

$$f_x(x) = \frac{3}{2}e^{-3x}u(x) + ae^{4x}u(-x)$$

where $u(x)$ is the unit step function. Then the value of 'a' and Prob $\{X \le 0\}$, respectively, are

(a) $2, \frac{1}{2}$
(b) $4, \frac{1}{2}$
(c) $2, \frac{1}{4}$
(d) $4, \frac{1}{4}$

[EE, 2016 : 2 Marks, Set-2]

Solution: (a)

$$f_x(x) = \begin{cases} ae^{4x} & x < 0 \\ \frac{3}{2}e^{-3x} & x \geq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f_x(x) = 1$$

$$\int_{-\infty}^{0} ae^{4x}dx + \int_{0}^{\infty}\frac{3}{2}e^{-3x}dx = 1$$

$$\left[\frac{ae^{4x}}{4}\right]_{-\infty}^{0} + \left[\frac{\frac{3}{2}e^{-3x}}{-3}\right]_{0}^{\infty} = 1$$

$$\frac{a}{4} + \frac{3}{6} = 1$$

$$a = 2$$

$$P(x \leq 0) = \int_{-\infty}^{0} 2e^{4x}dx = \left[\frac{e^{4x}}{2}\right]_{-\infty}^{0} = \frac{1}{2}$$

5.3.3.8 Normal Distribution

We say that X is a normal random variable, or simply that X is normally distributed, with parameters μ and σ^2, if the probability density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

The density function is a bell-shaped curve that is symmetric about μ.

For Normal Distribution:

$$\text{Mean} = E(X) = \mu$$

$$\text{Variance} = V(X) = \sigma^2$$

5.3.3.8.1 Standard Normal Distribution

Since the for N (μ, σ^2) varies with μ & σ^2 & the integral can only be evaluated numerically, it is more reasonable to reduce this distribution to another distribution called Standard normal distribution N (0, 1) for which, the shape & hence the integral values remain constant.

Since all N (μ, σ^2) problems can be reduced to N(0, 1) problems, we need only to consult a standard table giving calculations of area under N (0, 1) from 0 to any value of z.

The conversion from N (μ, σ^2) to N (0, 1) is effected by the following transformation.

$$Z = \frac{X-\mu}{\sigma}$$

Where Z is called standard normal variate.

For Standard Normal distribution:

$$\text{Mean} = E(X) = 0$$
$$\text{Variance} = V(X) = 1$$

Hence the standard normal distribution is also referred to as the N(0, 1) distribution.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.107 A class of first year B. Tech. students is composed of four batches A, B, C and D, each consisting of 30 students. It is found that the sessional marks of students in Engineering Drawing in batch C have a mean of 6.6 and standard deviation of 2.3. The mean and standard deviation of the marks for the entire class are 5.5 and 4.2, respectively. It is decided by the course instructor to normalize the marks of the students of all batches to have the same mean and standard deviation as that of the entire class. Due to this, the marks of a student in batch C are changed from 8.5 to

(a) 6.0
(b) 7.0
(c) 8.0
(d) 9.0

[CE, GATE-2006, 2 marks]

Solution: (d)

Let the mean and standard deviation of the students of batch C be μ_c and σ_c respectively, and the mean and standard deviation of entire class of first year students be μ and σ respectively.

Now given,
$$\mu_c = 6.6$$
$$\sigma_c = 2.3$$
and
$$\mu = 5.5$$
$$\sigma = 4.2$$

In order to normalise batch C to entire class, the normalised score (z scores) must be equated.

since
$$Z = \frac{x-\mu}{\sigma} = \frac{x-5.5}{4.2}$$
$$Z_c = \frac{x_c-\mu_c}{\sigma_c} = \frac{8.5-6.6}{2.3}$$

Equating these two and solving, we get
$$\frac{8.5-6.6}{2.3} = \frac{x-5.5}{4.2}$$
$$\Rightarrow \quad x = 8.969 \simeq 9.0$$

Q.108 The standard normal cumulative probability function (probability from $-\infty$ to x_n) can be approximated as

$$F(x_n) = \frac{1}{1+\exp(-1.7255\, x_n\, |x_n|^{0.12})}$$

where x_n = standard normal deviate. If mean and standard deviation of annual precipitation are 102 cm and 27 cm respectively, the probability that the annual precipitation will be between 90 cm and 102 cm is

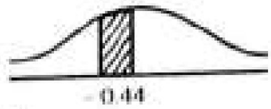
(a) 66.7%
(b) 50.0%
(c) 33.3%
(d) 16.7%

[CE, GATE-2009, 2 marks]

Solution: (b)

Here
$$\mu = 102\text{ cm}; \qquad \sigma = 27\text{ cm}$$
$$p(90 \le x \le 102) = p\left(\frac{90-102}{27} \le x_n \le \frac{102-102}{27}\right) = p(-0.44 \le x_n \le 0)$$

This area is shown below:



The shades area in above figure is

given by $$F(0) - F(-0.44) = \frac{1}{1+\exp(0)} - \frac{1}{1+\exp(-1.7255(-0.44)(0.44)0.12)}$$

$$= 0.5 - 0.3345 = 0.1655 \simeq 16.55\%$$

closest answer is 16.7%.

Q.109 The annual precipitation data of a city is normally distributed with mean and standard deviation as 1000 mm and 200 mm, respectively. The probability that the annual precipitation will be more than 1200 mm is

(a) < 50%
(b) 50%
(c) 75%
(d) 100% **[CE, GATE-2012, 1 mark]**

Solution: (a)

The annual precipitation is normally distributed with $\mu = 1000$ mm and $\sigma = 200$ mm

$$p(x > 1200) = p\left(z > \frac{1200-1000}{200}\right) = p(z > 1)$$

Where z is the standard normal variate.

In normal distribution

Now, since $p(-1 < z < 1) = 0.68$

($\simeq$ 68% of data is within one standard deviation of mean)

$$p(0 < z < 1) = \frac{0.68}{2} = 0.34$$

So $$p(z > 1) = 0.5 - 0.34 = 0.16 \simeq 16\%$$

Which is <50%

So choice (a) is correct.

Q.110 If {x} is a continuous, real valued random variable defined over the interval $(-\infty, +\infty)$ and its occurrence is defined by the density function given as:

$f(x) = \frac{1}{\sqrt{2\pi}\cdot b} e^{\frac{-1}{2}\left(\frac{x-a}{b}\right)^2}$ where 'a' and 'b' are the statistical attributes of the random variable {x}. The value of the integral $\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\cdot b} e^{\frac{-1}{2}\left(\frac{x-a}{b}\right)^2} dx$

(a) 1
(b) 0.5
(c) π
(d) $\pi/2$ **[CE, GATE-2014 : 1 Mark, Set-2]**

Solution : (b)

In normal distribution, the area under the normal curve from $-\infty$ to the mean = 0.5

Here, 'a' is the mean. So, The value of the integral

$$\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\cdot b} e^{\frac{-1}{2}\left(\frac{x-a}{b}\right)^2} dx = \text{the area under the normal curve from } -\infty \text{ to the mean} = 0.5$$

Q.111 Let X be a normal random variable with mean 1 and variance 4. The probability $P(X < 0)$ is

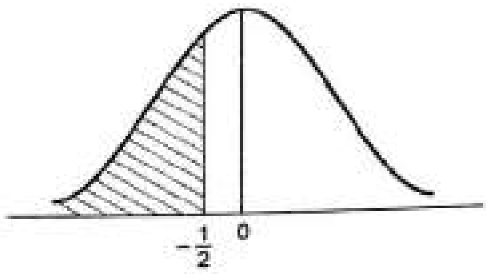
(a) 0.5
(b) greater than zero and less than 0.5
(c) greater than 0.5 and less than 1.0
(d) 1.0

[ME, GATE-2013, 1 Mark]

Solution: (b)

Here, $\sigma^2 = 4 \quad \Rightarrow \sigma = 2$

$$P(x < 0) = p\left(z < \frac{0-\mu}{\sigma}\right) = p\left(z < \frac{0-1}{2}\right) = p\left(z < -\frac{1}{2}\right)$$



Which is the shaded area in the picture and its value is clearly between 0. and 0.5

Q.112 A nationalized bank has found that the daily balance available in its savings accounts follows a normal distribution with a mean of Rs. 500 and a standard deviation of Rs. 50. The percentage of savings account holders, who maintain an average daily balance more than Rs. 500 is ____

[ME, GATE-2014 : 1 Mark, Set-1]

Solution :

Given X is normally distributed,

Given, $\mu = 500, \sigma = 50 p(x > 500) = p\left(z > \frac{500-\mu}{\sigma}\right) = p\left(z > \frac{500-500}{50}\right) = p(z > 0) = 0.5$

which is equal to 50%.

Q.113 The value of the integral $I = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \exp\left(-\frac{x^2}{8}\right)dx$ is

(a) 1
(b) π
(c) 2
(d) 2π

[EC, GATE-2005, 2 marks]

Solution: (a)

$$I = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{(-x^2/8)}dx$$

Comparing with $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1$

We can put μ and σ as any thing:

Here, putting $\mu = 0$

$$2\int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} = 1$$

Putting,
$$-\frac{x^2}{8} = -\frac{x^2}{2\sigma^2}$$

$\Rightarrow$
$$\sigma = 2.$$

Now putting $\sigma = 2$, in above equation, we get,

$$\therefore \quad \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{8}} = 1$$

Q.114 Let X be a randon variable following normal distribution with mean +1 and variance 4. Let Y be another normal variable with mean −1 and variance unknown,. If $P(X \le -1) = P(Y \ge 2)$ the standard deviation of Y is

(a) 3
(b) 2
(c) $\sqrt{2}$
(d) 1

[CS, GATE-2008, 2 marks]

Solution: (a)

Given, $\mu_X = 1$, $\sigma_x^2 = 4 \Rightarrow \sigma_x = 2$

Also given, $\mu_Y = -1$ and σ_Y is unknown

given, $p(X \le -1) = p(Y \ge 2)$

Converting into standard normal variates,

$$p\left(z \le \frac{-1-\mu_x}{\sigma_x}\right) = p\left(z \ge \frac{2-\mu_y}{\sigma_y}\right)$$

$$p\left(z \le \frac{-1-1}{2}\right) = p\left(z \ge \frac{2-(-1)}{\sigma_y}\right)$$

$$p(z \le -1) = p\left(z \ge \frac{3}{\sigma_y}\right) \quad \text{... (i)}$$

Now since us know that in standard normal distribution,

$$p(z \le -1) = p(z \ge 1) \quad \text{... (ii)}$$

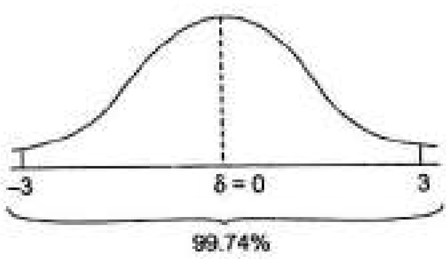
Comparing (i) and (ii) we can say that

$$\frac{3}{\sigma_y} = 1 \Rightarrow \sigma_y = 3$$

Q.115 The area (in percentage) under standard normal distribution curve of random variable *Z* within limits from −3 to +3 is______

[ME, 2016 : 1 Mark, Set-3]

Solution:



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CHAPTER 6

Numerical Methods

6.1 INTRODUCTION

Mathematical methods used to solve equations or evaluate integrals or solve differential equations can be classified broadly into two types.

1. Analytical Methods
2. Numerical Methods

6.1.1 Analytical Methods

Analytical methods an those which by an analysis of the equation obtain a solution directly as a readymade formulae in terms of say, the coefficients present in the equations.

Example: 1

Solve $ax^2 + bx + c$ analytically

Analytical solution: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: 2

Evaluate $\int x^2\, dx$ analytically

Analytical solution: $$\int_1^2 x^2\, dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3 - 1^3}{3} = \frac{7}{3}$$

Example: 3

Solve the differential equation

$\frac{dy}{dx} - 2y = 0$ with initial condition $y(0) = 3$.

Analytical solution: $$\int \frac{dy}{y} = \int 2\, dx$$

$$\Rightarrow \quad \log y = 2x$$

$$y = c\, e^{2x}$$

$$y(0) = 3$$

$$\Rightarrow \quad c = 3$$

$\therefore$ $y = 3e^{2x}$ is the required analytical solution.

6.1.2 Numerical Methods:

Those same problems could also be solved numerically as we shall see in this chapter.
In numerical solution, instead of directly writing the answer in terms of some formulae, we perform stepwise calculations using some algorithms or numerical procedures (usually on a computer) and arrive at the same results.
The advantage of numerical methods is that usually these procedures work on a much wider range of problems as compared to analytical solutions which work only on a limited class of problems.
For example, there are no analytical solutions available for polynomials of degree 4 or more. Whereas numerical methods can be used to solve polynomial equations of any degree.
Also numerical solutions can be used on linear as well as nonlinear equations, whereas analytical solutions usually fail for nonlinear equations.
With the advent of computers and huge computational (number crunching) power, numerical methods have largely replaced analytical methods of solution and have extended the power of mathematical methods to solving a much wider class of practical problems which occur in simulation and modeling, than it was possible before using analytical methods only.
Although Numerical Methods exist to solve so many types of commonly occurring mathematical problems, we shall focus on four problems in particular in this book, where numerical methods are successfully applied.

1. Solution of system of linear equations
2. Solution of algebraic and transcendental equations in single variable
3. Evaluation of definite integrals
4. Solution of ordinary differential equations

The advantage of numerical methods is its applicability to a wider class of mathematical problems, a disadvantage of numerical methods is that these methods introduce errors in varying degrees into the solution, thereby making them approximate. These errors however, can be controlled and contained within some ordinary tolerance local.

6.1.3 Errors in Numerical Methods

1. **Round-off Error:** It occurs due to limited storage space available inside computer for storing mantissa part of a floating point number due to which these numbers are either chopped off or rounded after so many significant digits.
2. **Truncation Error:** It occurs due to usage of fixed or limited number of terms of an infinite series to approximate certain functions.

Example:

Taylor's and Mclaurin's Series expansions of functions like e^x, sinx, cosx etc., with limited number of terms of the infinite series.

Although errors are introduced in Numerical Methods, they can be controlled and hence either reduced to arbitrarily low values or managed to be within tolerable limits.
For example, round-off errors can be controlled by allocating larger storage space for mantissa by using double float, instead of float for example.
Truncation errors can be controlled by developing methods in which more terms of the Taylor's series are used.
For example, truncation error in Simpson's rule of numerical integration is much less than trapezoidal rule for same problem, owing to the fact that Simpson's rule is developed by taking more terms of Taylor's Series. The order of a Numerical Method is a way of quantifying the extent of error, the higher

the order, lesser the error. Some numerical methods involve starting the procedure by assuming trial guess values for the solution and then refining the answer successively to greater and greater accuracy in each iteration. These types of numerical methods are called trial and error methods or iterative methods..

For example, the Gauss-Seidel method for solving system of linear equations is a trial and error (iterative) method. So is the bisection, regula-falsi, secant and Newton-Raphson methods used for root finding (solving algebraic and transcendental equations of the form f(x) = 0).

Quantifying Errors in Numerical Methods: There are several measures to quantity the error which occurs in numerical methods.

$$\text{Error} = \text{Exact Value} - \text{Approximate Value}$$

$$\text{Absolute Error} = |\text{Exact Value} - \text{Approximate Value}|$$

$$\text{Relative Error} = \left|\frac{\text{Exact} - \text{Approximate}}{\text{Exact}}\right|$$

$$\text{Relative Error \%} = \left|\frac{\text{Exact} - \text{Approximate}}{\text{Exact}}\right| \times 100$$

6.2 NUMERICAL SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Consider the following m first degree equations consisting of n unknowns $x_1, x_2 \ldots x_n$.

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1j} x_j + \ldots\ldots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2j} x_j + \ldots\ldots + a_{2n} x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mj}x_j + \ldots\ldots + a_{mn}x_{mn} = b_m$$

or in matrix notation, we have

$$\begin{bmatrix} a_{11} a_{12} \ldots\ldots a_{1n} \\ a_{21} a_{22} \ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots \\ a_{m1} a_{m2} \ldots\ldots a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}_{m \times 1}$$

$$\Rightarrow \quad AX = B$$

By finding a solution of the above system of equation we mean to obtain the values of $x_1, x_2 \ldots x_n$ such that they satisfy all the given equations simultaneously. The system of equations, given above is said to be homogenous if all b_i (i = 1 .... m) vanish, otherwise it is called as non homogenous system. There are number of methods to solve the above **System of Linear Equations**.

These are as follows:

1. Matrix Inversion Method
2. Cramer's Rule
3. Crout's and Dolittle's Method (Triangularisation Methods)
4. Gauss-Elimination Method
5. Gauss-Jordan's Method
6. Gauss-Seidel Iterative Method
7. Jacobi Iterative Method

In this book, we shall focus on Triangularisation, Gauss-Elimination and Gauss-Seidel Methods only.

6.2.1 Method of Factorisation or Triangularisation Method (Dolittle's Triangularisation Method)

This method is based on the fact that a square matrix A can be factorised into the form LU where L is unit lower triangular and U is a upper triangular, if all the principal minors of A are non singular i.e., it is a standard result of linear algebra that such a factorisation, when it exists, is unique.

We consider, for definiteness, the linear system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written in the form

$$AX = B \quad \text{... (i)}$$

Let $$A = LU \quad \text{... (ii)}$$

where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{... (iii)}$$

and $$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \text{... (iv)}$$

(i) becomes $$LUX = B \quad \text{... (v)}$$

If we set $$UX = Y \quad \text{... (vi)}$$

then (v) way be written as $$LY = B \quad \text{... (vii)}$$

which is equivalent to the system

$$y_1 = b_1$$
$$\ell_{21}y_1 + y_2 = b_2$$
$$\ell_{31}y_1 + \ell_{32}y_2 + y_3 = b_3$$

and can be solved for y_1, y_2, y_3 by the forward substitution. Once, Y is known, the system (vi) become

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3$$

which can be solved by backward substitution.

We shall now describe a scheme for computing the matrices L and U, and illustrate the procedure with a matrix of order 3. From the relation (ii), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the watrices on the left and equating the corresponding claments of both sides we get

$$u_{11} = a_{11},\ u_{12} = a_{12},\ u_{13} = a_{13}$$
$$\ell_{21}u_{11} = a_{21}$$

or $$\ell_{21} = \frac{a_{21}}{u_{11}},$$

$$\ell_{21}u_{12} + u_{22} = a_{22}$$

$\Rightarrow$ $$u_{22} = a_{22} - \ell_{21}u_{12}$$

$$\ell_{31}u_{13} + u_{23} = a_{23}$$

$$\Rightarrow \quad u_{23} = a_{23} - \ell_{21}u_{13}$$

$$\ell_{31}u_{11} = a_{31}$$

$$\Rightarrow \quad \ell_{31} = \frac{a_{31}}{u_{11}}$$

$$\ell_{31}u_{12} = \ell_{32}u_{22} = a_{32}$$

$$\Rightarrow \quad \ell_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}$$

Lastly, $\quad \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} = a_{33}$

$$\Rightarrow \quad u_{33} = a_{33} - \ell_{31}u_{13} - \ell_{32}u_{23}$$

∴ the variables are solved in the following order u_{11}, u_{12}, u_{13}

then $\ell_{21}, u_{22}, u_{23}$

lastly, $\ell_{31}, \ell_{32}, u_{33}$

ILLUSTRATIVE EXAMPLES

Example:

Solve the equations

$$2x + 3y + z = 9,$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

by the factorisation method.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

clearly $\quad u_{11} = 2, u_{12} = 3, u_{13} = 1$

also $\quad \ell_{21}u_{11} = 1$, so that $\ell_{21} = 1/2$

$$\ell_{21}u_{12} + u_{22} = 2$$

$$\Rightarrow \quad u_{22} = 2 - \ell_{21}u_{12} = 1/2$$

$$\ell_{21}u_{13} + u_{23} = 3$$

from which we obtain $\quad u_{23} = 5/2$

$$\ell_{31}u_{11} = 3$$

$$\Rightarrow \quad \ell_{31} = 3/2$$

$$\ell_{31}u_{12} + \ell_{32}u_{22} = 1$$

$$\Rightarrow \quad \ell_{32} = -7$$

$$\ell_{31}u_{13} + \ell_{32}u_{23} + u_{23} = 2$$

$$\Rightarrow \quad u_{33} = 18$$

It follows that
$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

and hence the given system of equations can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

or as
$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

solving this system by forward substitution, we get

$$y_1 = 9, \ \frac{y_1}{2} + y_2 = 6$$

$$\Rightarrow \qquad y_2 = \frac{3}{2}$$

$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \text{ or } y_3 = 5$$

Hence the solution of the original system is given by

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

which when solved by back substitution process.

$$x = \frac{35}{18}; \ y = \frac{29}{18}; \ z = \frac{5}{18}$$

Note: The Crout's triangularisation method is very similar to Dolittle's method except that in crout's method the $L = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$.

Also the order of solving the unknowns in crout's method is column wise instead of row wise i.e., we solve first $\ell_{11}, \ell_{21}, \ell_{31}$ then $u_{12}, \ell_{22}, \ell_{32}$ then u_{13}, u_{23} and ℓ_{33}. There is no particular advantage of crout's method over Dolittle's method and hence either method can be used for triangularisation.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 The matrix $[A]=\begin{bmatrix}2 & 1\\4 & -1\end{bmatrix}$ is decomposed into a product of a lower triangular matrix x[L] and an upper triangular matrix [U]. The properly decomposed [L] and [U] matrices respectively are

(a) $\begin{bmatrix}1 & 0\\4 & -1\end{bmatrix}$ and $\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix}$

(b) $\begin{bmatrix}2 & 0\\4 & -1\end{bmatrix}$ and $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 0\\4 & 1\end{bmatrix}$ and $\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$

(d) $\begin{bmatrix}2 & 0\\4 & -3\end{bmatrix}$ and $\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}$

[EE, GATE-2011, 2 marks]

Solution: (d)

Let us try Dolittle's decomposition by putting $l_{11} = 1$ & $l_{22} = 1$

$$\begin{bmatrix}2 & 1\\4 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\l_{21} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12}\\0 & u_{22}\end{bmatrix}$$

$$u_{11} = 2,\ u_{12} = 1$$

$$l_{21}u_{11} = 4$$

$$\Rightarrow \quad l_{21} = \frac{4}{2} = 2$$

$$l_{21}u_{12} + u_{22} = -1$$

$$\Rightarrow \quad 2 \times 1 + u_{22} = -1$$

$$\Rightarrow \quad u_{22} = -3$$

So one possible breakdown is

$$\begin{bmatrix}2 & 1\\4 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\0 & -3\end{bmatrix}$$

But this is not any of the choices given.

So let us do Crout's decomposition, by putting $u_{11} = 1$ and $u_{22} = 1$

$$\begin{bmatrix}2 & 1\\4 & -1\end{bmatrix} = \begin{bmatrix}l_{11} & 0\\l_{21} & l_{22}\end{bmatrix}\begin{bmatrix}1 & u_{12}\\0 & 1\end{bmatrix}$$

$$l_{11} = 2,\ l_{11}u_{12} = 1$$

$$\Rightarrow \quad u_{12} = \frac{1}{2} = 0.5$$

$$l_{21} = 4,\ l_{21}u_{12} + l_{22} = -1$$

$$\Rightarrow \quad 4 \times \frac{1}{2} + l_{22} = -1$$

$$\Rightarrow \quad l_{22} = -3$$

So $$\begin{bmatrix}2 & 1\\4 & -1\end{bmatrix} = \begin{bmatrix}2 & 0\\4 & -3\end{bmatrix}\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}$$

Which is choice (d).

Q.2 Match the application to appropriate numerical method.

Application

P1: Numerical integration

P2: Solution to a transcendental equation

P3: Solution to a system of linear equations

P4: Solution to a differential equation

M1: Newton-Raphson Method

M2: Runge-Kutta Method

M3: Simpson's 1/3-rule

M4: Gauss Elimination Method

(a) P1—M3, P2—M2, P3—M4, P4—M1

(b) P1—M3, P2—M1, P3—M4, P4—M2

(c) P1—M4, P2—M1, P3—M3, P4—M2

(d) P1—M2, P2—M1, P3—M3, P4—M4

[EC, GATE-2014 : 1 Mark, Set-3]

Answer : (b)

Q.3 In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1, then the lower diagonal entry l_{22} of L is ________.

[CS, GATE-2015 : 1 Mark, Set-1]

Solution: (5)

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$$

This is crout's LU decomposition, since diagonal elements of U are 1. So we will setup the equations for the elements of the matrix taken column-wise, as follows

$L_{11} = 2,\ L_{21} = 4$

$L_{11} \times U_{12} = 2 \Rightarrow U_{12} = 1,\ L_{21} \times U_{12} + L_{22} = 9 \Rightarrow 4 + L_{22} = 9 \Rightarrow L_{22} = 5$

6.2.2 Gauss Seidel Method

In the first equation of (ii), we substitute the first approximation

$(x_1^{(1)}, x_2^{(1)}, \ldots\ldots\ldots\ldots x_n^{(1)})$ into right hand side and denote the result as $x_1^{(2)}$.

In the second equation we substitute $(x_1^{(2)}, x_2^{(1)}, \ldots\ldots\ldots\ldots x_n^{(1)})$ and denote the result as $x_2^{(2)}$.

In the third approximation we substitute $(x_1^{(2)}, x_2^{(2)}, x_3^{(1)} \ldots\ldots\ldots\ldots x_n^{(1)})$ and call the result as $x_3^{(2)}$. In this manner, we complete the first stage of iteration and the entire process is repeated till the values of $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ are obtained to the accuracy required. It is clear therefore that this method uses an improved component as soon as it is available and it is called the method of **"Successive displacements"** or **"Gauss-Seidel method"**.

Note: It can be shown that the Gauss-Seidel method converges twice as fast as the **"Jacobi method"**.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.4 Gauss seidel method is used to solve the following equations (as per the given order):

$x_1 + 2x_2 + 3x_3 = 1 \quad ; \quad 2x_1 + 3x_2 + x_3 = 1 \quad ; \quad 3x_1 + 2x_2 + x_3 = 1$

Assuming initial guess as $x_1 = x_2 = x_3 = 0$, the value of x_3 after the first iteration is ____.

[ME, 2016 : 2 Marks, Set-1]

Solution:

The equations are

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 3x_2 + x_3 = 1 \quad ...(3)$$
$$3x_1 + 2x_2 + x_3 = 3 \quad ...(2)$$

By poveting we can write ...(1)

$$3x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 1$$
$$x_1 = \frac{3 - 2x_2 - x_3}{3} \quad ...(1)$$
$$x_2 = \frac{1 - 2x_1 - x_3}{3} \quad ...(2)$$
$$x_1 + 2x_2 + 3x_3 = 5$$
$$x_3 = \frac{5 - x_1 - 2x_2}{3} \quad ...(3)$$

Put $x_2 = 0$ $x_3 = 0$ in equation (1) $x_1 = 1$

Put $x_1 = 1$ $x_3 = 0$ in equation (3) $x_2 = -0.333$

Put $x_1 = 1$ $x_2 = -0.333$ in equation (3) $x_3 = 1.555$

$x_3 = 1.555$

6.3 NUMERICAL SOLUTIONS OF NONLINEAR ALGEBRAIC AND TRANSCENDENTAL EQUATIONS BY BISECTION, REGULA-FALSI, SECANT AND NEWTON-RAPHSON METHODS

In scientific and engineering work, a frequently occuring problem is to find the roots of equations of the form

$$f(x) = 0 \quad ...(i)$$

If f(x) is a quadratic, cubic or biquadratic expression then algebraic formula are available for expressing the roots in terms of the coefficients. On the other hand when f(x) is a polynomial of higher degree or on expression involving transcendental functions e.g., $1 + \cos x - 5x$, $x\tan x - \cosh x$, $e^x - \sin x$ etc. Algebraic methods are not available and recourse must be taken to find the roots by approximate methods.

There are some numerical methods for the solutions of equations of the form (1), where f(x) is algebraic or transcendental or a combinations of both.

6.3.1 Roots of Algebraic Equations

Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_{n-1}x + p_n$ be a rational integral function of x of n dimensions, and let us denote it by f(x); then f(x)= 0 is the general type of a rational integral equation of the n^{th} degree.

Dividing throughout by p_0, we see that without any loss of generality we may take

$$x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_{n-1}x + p_n = 0$$

as the type of a rational integral equation of n^{th} degree.

1. Unless otherwise stated the coefficients $p_1, p_2, ... p_n$ will always be supposed rational.
2. Any value of x which makes f(x) vanish is called a root of the equation f(x) = 0.
3. When f(x) is divided by x – a without remainder, a is a root of the equation f(x) = 0.
4. We shall assume that every equation of the form f(x) = 0 has a root, real or imaginary.
5. Every equation of the n^{th} degre has n roots, and no more.

Proof: Denote the given equation by f(x) = 0, where

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + ...+ p_n$$

The equation $f(x) = 0$ has a root, real or imaginary; let this be denoted by a_1; then $f(x)$ is divisible by $x - a_1$, so that

$$f(x) = (x - a_1)\,\phi_1(x)$$

where $\phi_1(x)$ is a rational integral function of $n - 1$ dimensions. Again, the equation $\phi_1(x) = 0$ has a root, real or imaginary; let this be denoted by a_2; then $\phi_1(x)$ is divisible by $x - a_2$, so that

$$\phi_1(x) = (x - a_2)\,\phi_2(x)$$

where $\phi_2(x)$ is a rational integral function of $n - 2$ dimensions.

Thus
$$f(x) = p_0(x - a_1)(x - a_2)\,\phi_2(x)$$

Proceeding in this way, we obtain,

$$f(x) = p_0(x - a_1)(x - a_2)\ldots\ldots(x - a_n).$$

Hence the equation $f(x) = 0$ has n roots, since $f(x)$ vanishes when x has any of the values $a_1, a_2, \ldots a_n$.

6. Also the equation cannot have more than n roots; for if x has any value different from any of the quantities $a_1, a_2, a_3, \ldots\ldots a_n$, all the factors on the right are different from zero, and therefore $f(x)$ cannot vanish for that value of x.

7. In the above investigation some of the quantities $a_1, a_2, a_3, \ldots a_n$ may be equal; in this case, however, we shall suppose that the equation has still n roots, although these are not all different.

8. In an equation with real coefficients imaginary roots occur in pairs.

   Suppose that $f(x) = 0$ is an equation with real coefficients, and suppose that it has an imaginary root $a + ib$; we shall show that $a - ib$ is also a roots. The factor $f(x)$ corresponding to these two roots is

   $$(x - a - ib)(x - a + ib), \text{ or } (x - a)^2 + b^2.$$

   Suppose that $a \pm ib$, $c \pm id$, $e \pm ig$, ... are the imaginary roots of the equation $f(x) = 0$, and that $f(x)$ is the product of the quadratic factors corresponding to these imaginary roots; then

   $$f(x) = \{(x - a)^2 + b^2\}\{(x - e)^2 + d^2\}\{(x - e)^2 + g^2\}\ldots.$$

   Now each of these factors is positive for every real value of x; hence $\phi(x)$ is always positive for real values of x.

9. We may show that in an equation with rational coefficients, surd roots enter in pairs; that is, if $a + \sqrt{b}$ is a root then $a - \sqrt{b}$ is also a root.

ILLUSTRATIVE EXAMPLES

Example:

Solve the equation $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, having given that one root is $2 - \sqrt{3}$.

Solution:

Since $2 - \sqrt{3}$ is a root, we know that $2 + \sqrt{3}$ is also a root, and corresponding to this pair of roots we have the quadratic factor $x^2 - 4x + 1$.

Also $6x^4 + 13x^3 - 35x^2 - x + 3 = (x^2 - 4x + 1)(6x^2 + 11x + 3)$;

hence the other roots are obtained from

$$6x^2 + 11x + 3 = 0,$$

or
$$(3x + 1)(2x + 3) = 0$$

thus the roots are $-\frac{1}{3}, -\frac{3}{2}, 2 + \sqrt{3}, 2 - \sqrt{3} = 0$

To determine the nature of some of the roots of an equation it is not always necessary to solve it; for instance, the truth of the following statements will be readily admitted.

1. If the coefficients are all positive, the equation has no positive root; thus the equation $x^6 + x^3 + 2x + 1 = 0$ cannot have a positive root.
2. If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the contarary sign, the equation has no negative roots; thus the equation $x^7 + x^5 - 2x^4 + x^3 - 3x^2 + 7x - 5 = 0$ cannot have a negative root
3. If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root; thus the equation $2x^8 + 3x^4 + x^2 + 7 = 0$ cannot have a real root.
4. If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except x = 0; thus the equation $x^9 + 2x^5 + 3x^3 + x = 0$ has no real root except x = 0.
All the foregoing results are included in the theorem of the next article, which is known as Descartes' Rule of Signs.

6.3.2 Descarte's Rule of Signs

An equation f(x) = 0 cannot have more positive roots than there are changes of sign in f(x), and cannot have more negative roots than there are changes of sign in f(–x).

i.e. number of real positive roots ≤ number of sign changes in f(x)

and number of real negative roots ≤ number of sign changes in f(–x).

ILLUSTRATIVE EXAMPLES

Example:

Consider the equation $x^9 + 5x^3 - x^3 + 7x + 2 = 0$.

Solution:

Here there are two changes of sign, therefore there are at most two positive roots.

Again $f(-x) = -x^9 + 5x^3 + x^3 - 7x + 2$, and here there are three changes of sign, therefore the given equation has at most three negative roots, and therefore it must have at least four imaginary roots, since total number of roots is nine, it being a ninth degree polynomial.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.5 Given that one root of the equation $x^3 - 10x^2 + 31x - 30 = 0$ is 5, the other two roots are

(a) 2 and 3
(b) 2 and 4
(c) 3 and 4
(d) –2 and –3 **[CE, GATE-2007, 2 marks]**

Solution: (a)

Since 5 is a root, f(x) is divisible by x – 5. Now dividing f(x) by x – 5 we get

$$x-5\overline{)x^3 - 10x^2 + 31x - 30}(x^2 - 5x + 6$$

$$\begin{array}{r} x^3 - 5x^2 \\ \hline -5x^2 + 31x - 30 \\ -5x^2 + 25x \\ \hline 6x - 30 \\ 6x - 30 \\ \hline 0 \\ \hline \end{array}$$

$\therefore \quad x^3 - 10x^2 + 31x - 30 = 0$

$\Rightarrow \quad (x-5)(x^2 - 5x + 6) = 0$

Roots of $x^2 - 5x + 6$ are 2 and 3.

∴ The other two roots are 2 and 3.

Q.6 If a continuous function $f(x)$ does not have a root in the interval $[a, b]$, then which one of the following statements is TRUE?

(a) $f(a) \cdot f(b) = 0$ (b) $f(a) \cdot f(b) < 0$
(c) $f(a) \cdot f(b) > 0$ (d) $f(a)/f(b) \le 0$

[EE, GATE-2015 : 1 Mark, Set-1]

Solution: (c)

Intermediate value theorem states that if a function is continuous and $f(a) \cdot f(b) < 0$, then surely there is a root in (a, b). The contrapositive of this theorem is that if a function is continuous and has no root in (a, b) then surely $f(a) \cdot f(b) \ge 0$. But since it is given that there is no root in the closed interval $[a, b]$ it means $f(a) \cdot f(b) \ne 0$.

So surely $f(a) \cdot f(b) > 0$ which is choice (c).

6.3.3 Numerical Methods for Root Finding

We shall study four numerical methods, all of which are iterative (trial and error methods) for root finding i.e. solving $f(x) = 0$.

1. Bisection Method
2. Regula-Falsi Method
3. Secant Method
4. Newton-Raphson Method

6.3.3.1 Bisection Method

This method is based on the intermediate value theorem which states that if a function f(x) is continuous between a and b, and f(a) and f(b) are of opposite signs then there exists at least one root between a and b for definiteness.

Let f(a) be negative, and f(b) be positive (see figure below). Then the root lies between a and b and let its approximate value be given by $x_0 = (a + b)/2$.

If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x_0) = 0$, otherwise the root lies either x_0 and b or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We designate this new interval as $[a_1, b_1]$ whose length is $|b - a|/2$.

As before this is bisected at x_1 and the new interval will be exactly half the length of the previous one. We repeat this process until the latest interval is as small as desired say ϵ. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the n^{th} step, the new interval will be $[a_n, b_n]$ of length $|b - a|/2^n$.

We then have $\frac{|b-a|}{2^n} \le \epsilon$ which gives on simplification $n \ge \frac{\log_e\left(\frac{|b-a|}{\epsilon}\right)}{\log_e 2}$ ...(i)

Inequality (i) gives the number of iterations required to achieve an accuracy ϵ.

This method can be shown graphically as follows

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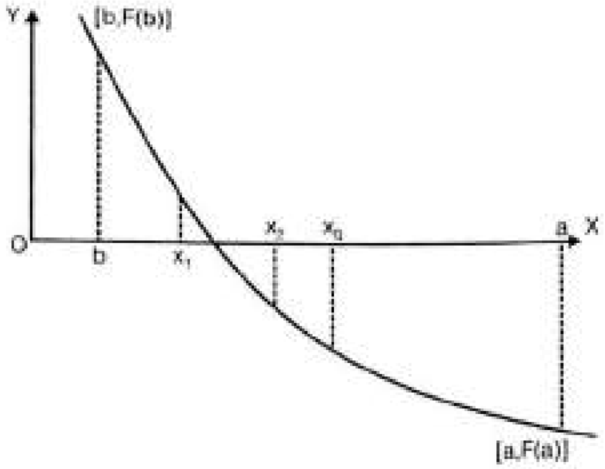


The iteration equation for binetion method is $x_2 = \frac{x_0 + x_1}{2}$ or more generally, $x_{n+1} = \frac{x_{n-1} + x_n}{2}$.

ILLUSTRATIVE EXAMPLES

Example:

Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution:

Since f(1) is negative and f(2) is positive, a root lies between 1 and 2 and therefore we take $x_0 = 3/2$.

Then $f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$ which is positive. Hence the root lies between 1 and 1.5 and we obtain $x_1 = (1 + 1.5)/2 = 1.25$ we find $f(x_1) = -19/64$, which is negative. We therefore, conclude that the root lies between 1.25 and 1.5. It follows that $x_2 = (1.25 + 1.5)/2 = 1.375$. The procedure is repeated and the successive approximations are, $x_3 = 1.3125$, $x_4 = 1.34375$, $x_5 = 1.328125$; etc.

6.3.3.2 Regula-Falsi Method

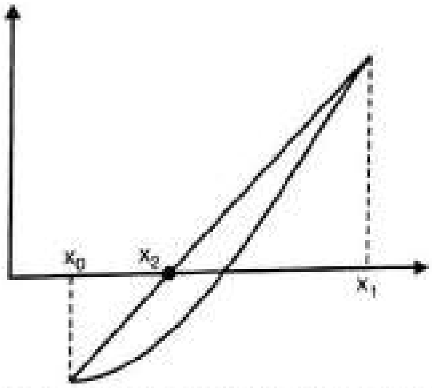
The method starts by taking two guess values x_0 and x_1 for the root, just like the bisection method, such that, $f(x_0)\, f(x_1) < 0$. The iteration formula for Regula-Falsi method is different from bisection method and it is

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

or more generally

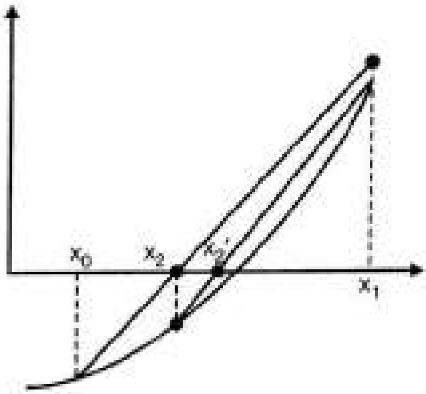
$$x_{n+1} = \frac{f_n x_{n-1} - f_{n-1} x_n}{f_n - f_{n-1}}$$

Graphically this can be shown as drawing a chord between (x_0, f_0) and (x_1, f_1) and seeing that the point of interaction of this chord with x axis is x_2, as shown below.



In the next iteration, the root is either between x_0 and x_2 or between x_1 and x_2.
So x_2 replaces either x_0 or x_1 depending on whether $f(x_0)\,f(x_2) < 0$ or $f(x_1)\,f(x_2 < 0)$.
If $f(x_0)\,f(x_2) < 0$ then x_1 replaced by x_2, else x_0 replaced by x_2. And the iteration is again continued and the new value of x'_2 in indicated by x_2 is figure below.

This is illustrated graphically as follows:



The process is continued until we get as close to the root as desired. Like bisection method, Regula-Falsi method is 100% reliable and the root will always be found, since always x_0 and x_1 are taken on either side of the root i.e. root is kept trapped between x_0 and x_1 in both bisection as well as Regula-Falsi methods.
Both Bisection and Regula-Falsi methods are (first order convergence or linear convergent), as compared with secant and Newton-Raphson methods which have convergence rates of 1.62 and 2 respectively i.e. Newton-Raphson method is quadratic convergent.

6.3.3.3 Secant Method

The Secant method proceeds similarly to Regula-Falsi method in the sense that it also requires two starting guess values, but the difference is that $f(x_0)\,f(x_1)$ need not be negative i.e. at any stage of iteration we do not ensure that the root is between x_0 and x_1. However, Secant method uses the same iteration equation as Regula-Falsi method.

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

or more generally

$$x_{n+1} = \frac{f_n x_{n-1} - f_{n-1} x_n}{f_n - f_{n-1}}$$

In Secant method, once the value of x_2 is obtained, to proceed to the next iteration, x_0 is always replaced by x_1 and x_1 is always replaced by x_2. This is the only and primary difference between Regula-Falsi and Secant method. Geometrically, both Regula-Falsi and Secant methods find x_2 by same way, that is by drawing the chord from (x_0, f_0) to (x_1, f_1) and intersection of this chord with x axis is x_2. The advantage of the Secant method is that it is faster than both the Bisection and Regula-Falsi method as it has a convergence order of 1.62. However, the disadvantage is that, Secant method is not 100% reliable, since the equation

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

will fail if $f_1 = f_0$, which may happen since no effort is made to keep f_1 and f_0 to be of opposite signs as it is done in case of Regula-Falsi method, which uses the same iteration equation.

6.3.3.4 Newton-Raphson Method

This method is generally used to improve the result obtained by one of the previous method. Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_0 + h)$ by Taylor's series we obtain

$$f(x) = f(x_0) + h\,f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0$$

Neglecting the second and higher order derivatives we have $f(x_0) + h\,f'(x_0) = 0$

which gives $$h = -\frac{f(x_0)}{f'(x_0)}$$

A better approximation than x_0 is therefore given by x_1, where

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are given by $x_2, x_3, \dots x_{n+1}$, where

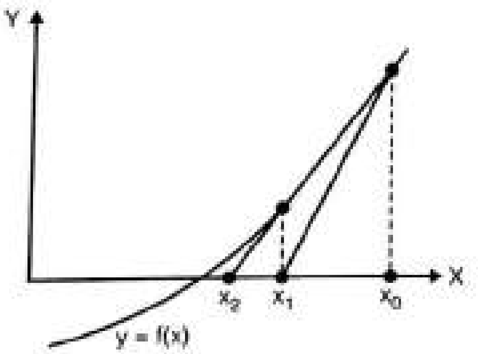
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \dots (i)$$

which is **Newton Raphson** formula.

$$\epsilon_{n+1} = -\frac{1}{2}\epsilon_n^2 \frac{f''(\xi)}{f'(\xi)} \quad \dots (ii)$$

So that the **Newton Raphson** process has a second order or quadratic convergence.

Geometrically, in Newton-Raphson method a tangent to curve is drawn at point $[x_0, f(x_0)]$ and the point of intersection of this tangent and x axis is taken as x_1 which is the next value of the iterate ofcourse x_1 is closer to root than x_0. It can be used for solving both algebraic and transcendental equations and it can also be used when the roots are complex.



The method converges rapidly to the root with a second order convergence. The number of significant digits in root which are correct, doubles, after each iteration of N-R method.

Following is a list of Common Newton Raphson iterative problems alongwith the Newton-Raphson iteration equation, for solving that problem.

1. The inverse of b, is the root of the equation $f(x) = \frac{1}{x} - b = 0$

   Iteration Equation: $$x_{n+1} = x_n(2 - bx_n)$$

2. The inverse square root b, is the root of equation $f(x) = \frac{1}{x^2} - b = 0$

Iteration Equation: $$x_{n+1} = \frac{1}{2}x_n\left(3 - bx_n^2\right)$$

3. The p^{th} root of a given number N, is root of equation $f(x) = x^p - N = 0$

Iteration Equation: $$x_{n+1} = \frac{(p-1)x_n^p + N}{px_n^{p-1}}$$

Note: The order of Bisection, Regular Falsi and Secant Method and Newton Raphson Method are given below:

Sl. no.	Method	Order
1.	Bisection	1
2.	Regula Falsi	1
3.	Secant Method	1.62
4.	Newton Raphson	2

ILLUSTRATIVE EXAMPLES FROM GATE

Statement for Linked Answer Questions 8 and 9.

Given $a > 0$, we wish to calculate the reciprocal value $\frac{1}{a}$ by Newton-Raphson method for $f(x) = 0$.

Q.7 The Newton Raphson algorithm for the function will be

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$

(b) $x_{k+1} = \left(x_k + \frac{a}{2}x_k^2\right)$

(c) $x_{k+1} = 2x_k - ax_k^2$

(d) $x_{k+1} = x_k - \frac{a}{2}x_k^2$

[CE, GATE-2005, 2 marks]

Solution: (c)

To calculate $\frac{1}{a}$ using N-R method,

set up the equation as $$x = \frac{1}{a}$$

i.e. $$\frac{1}{x} = a$$

$$\Rightarrow \quad \frac{1}{x} - a = 0$$

i.e. $$f(x) = \frac{1}{x} - a = 0$$

Now $$f'(x) = -\frac{1}{x^2}$$

$$f(x_k) = \frac{1}{x_k} - a$$

$$f'(x_k) = -\frac{1}{x_k^2}$$

For N-R method, $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$\Rightarrow$ $x_{k+1} = x_k - \frac{(1/x_k - a)}{-\frac{1}{x_k^2}}$

Simplifying which we get $x_{k+1} = 2x_k - ax_k^2$

Q.8 For a = 7 and starting with $x_0 = 0.2$, the first two iterations will be

(a) 0.11, 0.1299
(b) 0.12, 0.1392
(c) 0.12, 0.1416
(d) 0.13, 0.1428 **[CE, GATE-2005, 2 marks]**

Solution: (b)

For $a = 7$ the iteration equation,

becomes $x_{k+1} = 2x_k - 7x_k^2$

with $x_0 = 0.2$

$x_1 = 2x_0 - 7x_0^2 = 2 \times 0.2 - 7(0.2)^2 = 0.12$

and $x_2 = 2x_1 - 7x_1^2 = 2 \times 0.12 - 7(0.12)^2 = 0.1392$

Q.9 The following equation needs to be numerically solved using the Newton-Raphson method.

$$x^3 + 4x - 9 = 0$$

The iterative equation for this purpose is (k indicates the iteration level)

(a) $x_{k+1} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$
(b) $x_{k+1} = \frac{3x_k^2 + 4}{2x_k^2 + 9}$
(c) $x_{k+1} = x_k - 3x_k^2 + 4$
(d) $x_{k+1} = \frac{4x_k^2 + 3}{9x_k^2 + 2}$

[CE, GATE-2007, 2 marks]

Solution: (a)

$$f(x) = x^3 + 4x - 9 = 0$$

$$f'(x) = 3x^2 + 4$$

N-R equation for iteration is, $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$$f(x_k) = x_k^3 + 4x_k - 9$$

$$f'(x_k) = 3x_k^2 + 4$$

$$x_{k+1} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2 + 4)} = \frac{(3x_k^3 + 4x_k) - (x_k^3 + 4x_k - 9)}{3x_k^2 + 4}$$

$$x_{k+1} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$$

Q.10 The square root of a number N is to be obtained by applying the Newton Raphson iterations to the equation $x^2 - N = 0$. If i denotes the iteration index, the correct iterative scheme will be

(a) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$
(b) $x_{i+1} = \frac{1}{2}\left(x_i^2 + \frac{N}{x_i^2}\right)$
(c) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N^2}{x_i}\right)$
(d) $x_{i+1} = \frac{1}{2}\left(x_i - \frac{N}{x_i}\right)$

[CE, GATE-2011, 2 marks]

Solution: (a)

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \left(\frac{x_i^2 - N}{2x_i}\right) = \frac{x_i^2 + N}{2x_i} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Q.11 The quadratic equation $x^2 - 4x + 4 = 0$ is to be solved numerically, starting with the initial guess $x_0 = 3$. The Newton-Raphson method is applied once to get a new estimate and then the Secant method is applied once using the initial guess and this new estimate. The estimated value of the root after the application of the Secant method is ________.

[CE, GATE-2015 : 2 Marks, Set-I]

Solution:

$$f(x) = x^2 - 4x + 4$$
$$f'(x) = 2x - 4$$
$$x_0 = 3$$
$$f(3) = 1,\ f'(3) = 2$$

By Newton Rapshon method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{1}{2} = \frac{5}{2}$$

$$f\left(\frac{5}{2}\right) = \frac{25}{4} - 10 + 4 = \frac{1}{4}$$

By secant method,
$$x_2 = x_1 - \frac{(x_1 - x_0)f(x_1)}{f(x_1) - f(x_0)} = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0} = \frac{\left(\frac{1}{4}\times 3\right) - \left(1 \times \frac{5}{2}\right)}{\frac{1}{4} - 1} = \frac{7}{3}$$

Q.12 In Newton-Raphson iterative method, the initial guess value (x_{ini}) is considered as zero while finding the roots of the equation: $f(x) = -2 + 6x - 4x^2 + 0.5x^3$. The correction, Δx, to be added to x_{ini} in the first iteration is ________.

[CE, GATE-2015 : 1 Mark, Set-II]

Solution:

$$f(x) = -2 + 6x - 4x^2 + 0.5x^3$$
$$f'(x) = 6 - 8x + 1.5x^2$$
$$x_{ini} = 0$$

By Newton Raphson Method,
$$x_1 = x_{ini} - \frac{f(x_{ini})}{f'(x_{ini})} = 0 - \frac{-2}{6}$$

$\Rightarrow$
$$x_1 = \frac{1}{3}$$

$\therefore$
$$\Delta x = x_1 - x_{ini} = \frac{1}{3}$$

Q.13 Starting from $x_0 = 1$, one step of Newton-Raphson method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value (x_1) as

(a) $x_1 = 0.5$
(b) $x_1 = 1.406$
(c) $x_1 = 1.5$
(d) $x_1 = 2$

[ME, GATE-2005, 2 marks]

Solution: (c)

From Newton- Raphson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{...(i)}$$

Given function is $f(x) = x^3 + 3x - 7$

and $f'(x) = 3x^2 + 3$

Putting $x_0 = 1,$

$$f(x_0) = f(1) = (1)^3 + 3 \times (1) - 7 = -3$$

$$f'(x_0) = f'(1) = 3 \times (1)^2 + 3 = 6$$

Substituting x_0, $f(x_0)$ and $f'(x_0)$ values into (i) we get,

$$\therefore \quad x_1 = 1-\left(\frac{-3}{6}\right)\times 1 = 1.5$$

Q.14 The real root of the equation $5x - 2\cos x - 1 = 0$ (up to two decimal accuracy) is ______.

[ME, GATE-2014 : 2 Marks, Set-3]

Solution :

$$f(x) = 5x - 2\cos x - 1$$

$$f'(x) = 5 + 2\sin x$$

By Newton Raphson's equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Assuming $x_0 = 1$ (1 rad = 57.32°)

$$\Rightarrow \quad x_1 = 1-\frac{5\times 1-2\cos(57.32°)-1}{5+2\sin(57.32°)} = 0.5632$$

Again,

$$x_2 = 0.5632-\frac{5\times 0.5632-2\cos(32.27°)-1}{5+2\sin(32.27°)} = 0.5425$$

$$x_3 = 0.5425-\frac{5\times 0.5425-2\cos(31.09°)-1}{5+2\sin(31.09°)} = 0.5424$$

$\therefore$ Real root, $x = 0.54$

Q.15 Newton-Raphson method is used to find the roots of the equation, $x^3 + 2x^2 + 3x - 1 = 0$. If the initial guess is $x_0 = 1$, then the value of x after 2nd iteration is ________.

[ME, GATE-2015 : 2 Marks, Set-3]

Solution: (0.3043)

$$f(x) = x^3 + 2x^2 + 3x - 1$$

$$f'(x) = 3x^2 + 4x + 3$$

$$x_0 = 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1-\frac{[5]}{[10]} = 0.5$$

$$x_2 = 0.5-\frac{[f(0.5)]}{[f'(0.5)]} = 0.5-\frac{[0.125]}{[5.15]} = 0.3043$$

Q.16 Equation $e^x - 1 = 0$ is required to be solved using Newton's method with an initial guess $x_0 = -1$. Then, after one step of Newton's method, estimate x_1 of the solution will be given by

(a) 0.71828
(b) 0.36784
(c) 0.20587
(d) 0.00000

[EE, GATE-2008, 2 marks]

Solution: (a)

Here
$$f(x) = e^x - 1$$
$$f'(x) = e^x$$

The newton Raphson iterative equation is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
$$f(x_i) = e^{x_i} - 1$$
$$f'(x_i) = e^{x_i}$$

∴
$$x_{i+1} = x_i - \frac{e^{x_i} - 1}{e^{x_i}}$$

i.e.
$$x_{i+1} = \frac{x_i e^{x_i} - (e^{x_i} - 1)}{e^{x_i}} = \frac{e^{x_i}(x_i - 1) + 1}{e^{x_i}}$$

Now put $i = 0$
$$x_1 = \frac{e^{x_0}(x_0 - 1) + 1}{e^{x_0}}$$

Put $x_0 = -1$ as given,
$$x_1 = [e^{-1}(-2) + 1]/e^{-1} = 0.71828$$

Q.17 Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton-Raphson's method is given by

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$
(b) $x_{k+1} = x_k - \frac{117}{x_k}$
(c) $x_{k+1} = x_k - \frac{x_k}{117}$
(d) $x_{k+1} = x_k - \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$

[EE, GATE-2009, 2 marks]

Solution: (a)

$$x_{K+1} = x_K - \frac{f(x_K)}{f'(x_K)} = x_K - \frac{x_K^2 - 117}{2x_K} = \frac{1}{2}\left[x_K + \frac{117}{x_K}\right]$$

Q.18 Roots of the algebraic equation $x^3 + x^2 + x + 1 = 0$ are

(a) $(+1, +j, -j)$
(b) $(+1, -1, +1)$
(c) $(0, 0, 0)$
(d) $(-1, +j, -j)$

[EE, GATE-2011, 1 marks]

Solution: (d)

-1 is one of the roots since

$$(-1)^3 + (-1)^2 + (-1) + 1 = 0$$

By polynomial division

$$\frac{x^3 + x^2 + x + 1}{\{x - (-1)\}} = x^2 + 1$$

⇒
$$x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$$

So roots are $(-1, +j, -j)$

Q.19 Solution of the variables x_1 and x_2 for the following equations is to be obtained by employing the Newton-Raphson iterative method

equation (i) $10x_2 \sin x_1 - 0.8 = 0$

equation (ii) $10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$

Assuming the initial values $x_1 = 0.0$ and $x_2 = 1.0$, the Jacobian matrix is

(a) $\begin{bmatrix} 10 & -0.8 \\ 0 & -0.6 \end{bmatrix}$
(b) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -0.8 \\ 10 & -0.6 \end{bmatrix}$
(d) $\begin{bmatrix} 10 & 0 \\ 10 & -10 \end{bmatrix}$

[EE, GATE-2011, 2 marks]

Solution: (b)

$$u(x_1, x_2) = 10x_2 \sin x_1 - 0.8 = 0$$

$$v(x_1, x_2) = 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$$

The Jacobian matrix is

$$\begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 10x_2 \cos x_1 & 10 \sin x_1 \\ 10x_2 \sin x_1 & 20x_2 - 10 \cos x_1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Q.20 When the Newton-Raphson method is applied to solve the equation $f(x) = x^3 + 2x - 1 = 0$, the solution at the end of the first iteration with the initial guess value as $x_0 = 1.2$ is

(a) −0.82
(b) 0.49
(c) 0.705
(d) 1.69

[EE, GATE-2013, 2 Marks]

Solution: (c)

$$f'(x) = 3x^2 + 2$$

$$f'(x_0) = 3(1.2)^2 + 2 = 6.32$$

$$f(x_0) = (1.2)^3 + 2 \times 1.2 - 1 = 3.128$$

$$f'(x_1) = x_0 - \frac{f(x_0)}{f'(x_1)} = 1.2 - \frac{3.728}{6.32} = 0.705$$

Q.21 The function $f(x) = e^x - 1$ is to be solved using Newton-Raphson method. If the initial value of x_0 is taken as 1.0, then the absolute error observed at 2nd iteration is ______ .

[EE, GATE-2014 : 2 Marks, Set-3]

Solution :

Given, $f(x) = e^x - 1$

or, $f(x_K) = (e^{x_K} - 1)$ and $x_0 = 1$

In Newton-Raphson method, we have:

$$x_{K+1} = \left[x_K - \frac{f(x_K)}{f'(x_K)}\right]$$

$$\therefore \quad x_1 = \left[x_0 - \frac{f(x_0)}{f'(x_0)}\right] \quad \text{...(i)}$$

Now, $f(x_0) = e^{x_0} - 1 = e^1 - 1 = (e - 1)$

and $f'(x) = e^x$

$\therefore \quad f'(x_0) = e^1 = e$

Putting the values, we get:

$$x_1 = \left[1 - \frac{(e-1)}{e}\right] = \left[\frac{e - e + 1}{e}\right] = e^{-1} = 0.367$$

Also, $$x_2 = \left[x_1 - \frac{f(x_1)}{f'(x_1)}\right] \quad \text{...(ii)}$$

Now, $x_1 = \frac{1}{e} = e^{-1}$ and $f(x_1) = (e^{e^{-1}} - 1)$,

$$f'(x_1) = e^{e^{-1}}$$

Putting the values, we get:

$$x_2 = \left[e^{-1} - \frac{(e^{e^{-1}} - 1)}{e^{e^{-1}}}\right] = \left[e^{-1} - \frac{(e^{0.37} - 1)}{e^{0.37}}\right] = 0.06$$

Therefore, the absolute error observed at second iteration = 0.06.

Absolute error at any iteration

$$= \left|\frac{\text{Exact value} - \text{Approximate value}}{\text{Exact value}}\right| \simeq \left|\frac{\text{New value} - \text{Old value}}{\text{New value}}\right|$$

$$\simeq \left|\frac{0.06 - 0.368}{0.06}\right| = 0.248$$

Q.22 The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the Newton-Raphson method. If $x = 2$ is taken as the initial approximation of the solution, then the next approximation using this method will be

(a) $\frac{2}{3}$

(b) $\frac{4}{3}$

(c) 1

(d) $\frac{3}{2}$

[EC, GATE-2007, 2 marks]

Solution: (b)

Here,

$$x_0 = 2$$
$$f(x) = x^3 - x^2 + 4x - 4$$
$$f'(x) = 3x^2 - 2x + 4$$
$$f(x_0) = f(2) = 8$$
$$f'(x_0) = f'(2) = 12$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8}{12} = \frac{4}{3}$$

Q.23 The recursion relation to solve $x = e^{-x}$ using Newton-Raphson method is

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

(c) $x_{n+1} = (1+x_n)\dfrac{e^{-x_n}}{1+e^{-x_n}}$

(d) $x_{n+1} = \dfrac{x_n^2 - e^{-x_n}(1+x_n) - 1}{x_n - e^{-x_n}}$

[EC, GATE-2008, 2 marks]

Solution: (c)

The given equation to be solved is

$$x = e^{-x}$$

Which can be rewritten as

$$f(x) = x - e^{-x} = 0$$

$$f'(x) = 1 + e^{-x}$$

The Newton-Raphson iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Here

$$f(x_n) = x_n - e^{-x_n}$$

$$f'(x_n) = 1 + e^{-x_n}$$

∴ The Newton-Raphson iterative formula is

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1+e^{-x_n}} = \frac{e^{-x_n}x_n + e^{-x_n}}{1+e^{-x_n}} = (1+x_n)\frac{e^{-x_n}}{1+e^{-x_n}}$$

Q.24 A numerical solution of the equation $f(x) = x + \sqrt{x} - 3 = 0$ can be obtained using Newton-Raphson method. If the starting value is $x = 2$ for the iteration, the value of x that is to be used in the next step is

(a) 0.306

(b) 0.739

(c) 1.694

(d) 2.306

[EC, GATE-2011, 2 marks]

Solution: (c)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x = 2$,

$$f(x_0) = 2 + \sqrt{2} - 3 = \sqrt{2} - 1$$

$$f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$f'(x_0) = 1 + \frac{1}{2\sqrt{2}}$$

Then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{\sqrt{2}-1}{1+\frac{1}{2\sqrt{2}}}$$

⇒

$$x_1 = 1.694$$

Q.25 The Newton-Raphson method is used to solve the equation $f(x) = x^3 - 5x^2 + 6x - 8 = 0$. Taking the initial guess as $x = 5$, the solution obtained at the end of the first iteration is_____.

[EC, GATE-2015 : 2 Marks, Set-3]

Solution: (4.290)

$$f(x) = x^3 - 5x^2 + 6x - 8$$

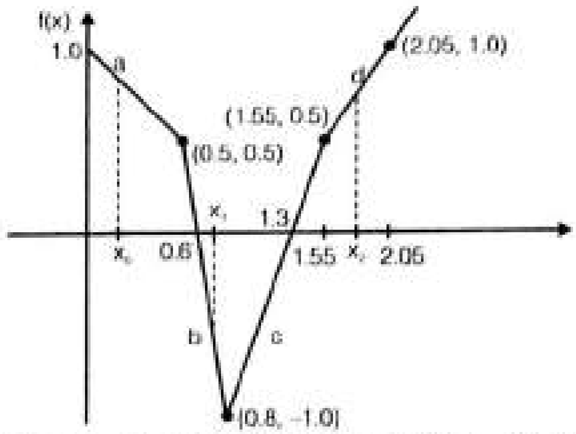
$$x_0 = 5$$

$$f'(x) = 3x^2 - 10x + 6$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{f(5)}{f'(5)}$$

$$= 5 - \frac{5^3 - 5 \times 5^2 + 6 \times 5 - 8}{3 \times 5^2 - 10 \times 5 + 6} = 5 - \frac{22}{31} = 5 - 0.7097 = 4.2903$$

Q.26 A piecewise linear function f(x) is plotted using thick solid lines in the figure below (the plot is drawn to scale).



If we use the Newton-Raphson method to find the roots of f(x) = 0 using x0, x1 and x2 respectively as initial guesses, the roots obtained would be

(a) 1.3, 0.6 and 0.6 respectively
(b) 0.6, 0.6 and 1.3 respectively
(c) 1.3, 1.3 and 0.6 respectively
(d) 1.3, 0.6 and 1.3 respectively

[CS, GATE-2003, 2 marks]

Solution: (d)

Starting from x_0, slope of line a $= \frac{1-0.5}{0-0.5} = -1$

y-intercept = 1

Eqn. of a is y = mx + c = −1x + 1

This line will cut x axis (i.e., y = 0), at x = 1

Since x = 1 is > than x = 0.8, a perpendicular at x = 1 will cut the line c and not line b.

∴ root will be 1.3

Starting from x_1,

the perpendicular at x_1 is cutting line b and root will be 0.6.

Starting from x_2,

$$\text{Slope of line d} = \frac{1-0.5}{2.05-1.55} = 1$$

Equation of d is $\quad y - 0.5 = 1(x - 1.55)$

i.e. $\quad y = x - 1.05$

This line will cut x axis at x = 1.05

Since, x = 1.05 is > than x = 0.8, the perpendicular at x = 1.05 will cut the line c and not line b. The root will be therefore equal to 1.3.

So starting from x_0, x_1 and x_2 the roots will be respectively 1.3, 0.6 and 1.3.

Q.27 Consider the series $x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}$, $x_0 = 0.5$ obtained from the Newton-Raphson method. The series converges to

(a) 1.5
(b) $\sqrt{2}$
(c) 1.6
(d) 1.4

[CS, GATE-2007, 2 marks]

Solution: (a)

Given, $$x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n},\ x_0 = 0.5$$

as $n \to \infty$, when the series converges

$$x_{n+1} = x_n = \alpha = \text{root of equation}$$

$$\alpha = \frac{\alpha}{2} + \frac{9}{8\alpha}$$

$$\alpha = \frac{4\alpha^2 + 9}{8\alpha}$$

$$\Rightarrow \quad 8\alpha^2 = 4\alpha^2 + 9$$

$$\Rightarrow \quad \alpha^2 = \frac{9}{4}$$

$$\alpha = \frac{3}{2} = 1.5$$

Q.28 The Newton-Raphson iteration $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ can be used to compute the

(a) square of R
(b) reciprocal of R
(c) square root of R
(d) logarithm of R

[CS, GATE-2008, 2 marks]

Solution: (c)

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

at convergence $$x_{n+1} = x_n = \alpha$$

$$\alpha = \frac{1}{2}\left(\alpha + \frac{R}{\alpha}\right)$$

$$2\alpha = \alpha + \frac{R}{\alpha} = \frac{\alpha^2 + R}{\alpha}$$

$$2\alpha^2 = \alpha^2 + R$$

$$\Rightarrow \quad \alpha^2 = R$$

$$\alpha = \sqrt{R}$$

So, this iteration will compute the square roof of R.
Correct choice is (c).

Q.29 Newton-Raphson method is used to compute a root of the equation $x^2 - 13 = 0$ with 3.5 as the initial value. The approximation after one iteration is

(a) 3.575
(b) 3.677
(c) 3.667
(d) 3.607

[CS, GATE-2010, 1 mark]

Solution: (d)

The equation is $f(x) = x^2 - 13 = 0$

Newton-Raphson iteration equation is

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right]$$

$$f(x_0) = x_0^2 - 13$$

$$f'(x_0) = 2x_0$$

$$\therefore \quad x_1 = x_0 - \left[\frac{x_0^2 - 13}{2x_0}\right] = \frac{x_0^2 + 13}{2x_0}$$

put $\quad x_0 = 3.5$ (as given)

$$x_1 = \frac{3.5^2 + 13}{2 \times 3.5} = 3.607$$

$\therefore$ The approximation after one interation = 3.607

Q.30 The bisection method is applied to compute a zero of the function $f(x) = x^4 - x^3 - x^2 - 4$ in the interval [1, 9]. The method converges to a solution after ____ iterations.

(a) 1
(b) 3
(c) 5
(d) 7

[CS, GATE-2012, 2 marks]

Solution: (b)

If bisection method is applied to given problem with $x_0 = 1$ and $x_1 = 9$

After 1 iteration $x_2 = \frac{1+9}{2} = 5$

Now since $f(x_1)\, f(x_2) > 0$, x_2 replaces x_1

Now, $x_0 = 1$ and $x_1 = 5$

and after 2nd iteration $x_2 = \frac{1+5}{2} = 3$

Now since $f(x_1)\, f(x_2) > 0$, x_2 replaces x_1 and $x_0 = 1$ and $x_1 = 3$ and after 3rd iteration

$$x_2 = \frac{1+3}{2} = 2$$

Now $f(x_2) = f(2) = 2^4 - 2^3 - 2^2 - 4 = 0$

So the method converges exactly to the root in 3 iterations.

Q.31 In the Newton-Raphson method, an initial guess of $x_0 = 2$ is made and the sequence $x_0, x_1, x_2 \ldots$ is obtained for the function

$$0.75x^3 - 2x^2 - 2x + 4 = 0$$

Consider the statements

(I) $x_3 = 0$.

(II) The method converges to a solution in a finite number of iterations.

Which of the following is TRUE?

(a) Only I
(b) Only II
(c) Both I and II
(d) Neither I nor II

[CS, GATE-2014 (Set-2) : 2 Marks]

Solution : (a)

Compute $x_1, x_2, \ldots$ using the iteration equation

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = x_0 - \left[\frac{0.75x_0^3 - 2x_0^2 - 2x_0 + 4}{2.25x_0^2 - 4x_0 - 2}\right]$$

$\Rightarrow x_0 = 2,\ x_1 = 0,\ x_2 = 2,\ x_3 = 0, \ldots$

$x_3 = 0$ is correct but it converges in an infinite steps (i.e. it doesn't converge).

Q.32 The secant method is used to find the root of an equation $f(x) = 0$. It is started from two distinct estimates x_a and x_b for the root. It is an iterative procedure involving linear interpolation to a root. The iteration stops if $f(x_b)$ is very small and then x_b is the solution. The procedure is given below. Observe that there is an expression which is missing and is marked by ?. Which is the suitable expression that is to be put in place of ? So that it follows all steps of the secant method?

```
Secant
Initialize: xa, xb, ε, N                // ε = convergence indicator
fb = f(xb)                              // N = maximum number of iterations
i = 0
while (i < N and | fb | > ε) do
        i = i + 1                       // update counter
        xt = ?                          // missing expression for
                                        // intermediate value
        xa = xb                         // reset xa
        xb = xt                         // reset xb
        fb = f(xb)                      // function value at new xb
end while
if |fb| > ε then                        // loop is terminated with i = N
        write "Non-convergence"
else
        write "return xb"
end if
```

(a) $x_b - (f_b - f(x_a))\, f_b / (x_b - x_a)$
(b) $x_a - (f_a - f(x_a))\, f_a / (x_b - x_a)$
(c) $x_b - (x_b - x_a)\, f_b / (f_b - f(x_a))$
(d) $x_a - (x_b - x_a)\, f_a / (f_b - f(x_a))$

[CS, GATE-2015 : 2 Marks, Set-2]

Solution: (c & d)

Secant method formula is $x_2 = \dfrac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$

i.e. $x_t = \dfrac{f_b x_a - f_a x_b}{f_b - f_a}$

$$x_t = x_b - (x_b - x_a)\, f_b / (f_b - f(x_a)) = x_a - (x_b - x_a)\, f_a / (f_b - f(x_a)) = \frac{f_b x_a - f_a x_b}{f_b - f_a}$$

$\therefore$ Both (c) & (d) after simplification reduce to the required formula. So both (c) and (d) are correct.

Q.33 Solve the equation $x = 10\cos(x)$ using the Newton-Raphson method. The initial guess is $x = \frac{\pi}{4}$. The value of the predicted root after the first iteration, up to second decimal, is________.

[ME, 2016 : 1 Mark, Set-1]

Solution:

$$f(x) = x - 10\cos x \quad f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{10}{\sqrt{2}} = -6.286$$

$$f'(x) = 1 + 10\sin x \quad f'\left(\frac{\pi}{4}\right) = 1 + \frac{10}{\sqrt{2}} = 8.07$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{\pi}{4} - \left(\frac{-6.286}{8.07}\right) = \frac{\pi}{4} + \frac{6.286}{8.07} = \mathbf{1.5639}$$

Q.34 The root of the function $f(x) = x^3 + x - 1$ obtained after first iteration on application of Newton Raphson scheme using an initial guess of $x_0 = 1$ is

(a) 0.682
(b) 0.686
(c) 0.750
(d) 1.000

[ME, 2016 : 1 Mark, Set-3]

Solution: (c)

$$f(x) = x^3 + x - 1 \qquad f(1) = 1 + 1 - 1 = 1$$

$$f'(x) = 3x^2 + 1 \qquad f'(1) = 3 + 1 = 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = 1 - 0.25 = \mathbf{0.750}$$

Q.35 Newton-Raphson method is to be used to find foot of equation $3x - e^x + \sin x = 0$. If the initial trail value of the roots is taken as 0.333, the next approximation for the root would be________.

[CE, 2016 : 1 Mark, Set-I]

Solution:

According to Newton-Raphson Method:

$$X_{N+1} = X_N - \frac{f(X_N)}{f'(X_N)}$$

$$f(x) = 3x - e^x + \sin x$$

$$f'(x) = 3 - e^x + \cos x$$

$$\Rightarrow \qquad X_1 = X_0 - \frac{f(0.333)}{f'(0.333)}$$

$$= 0.333 - \frac{3 \times 0.333 - e^{0.333} + \sin 0.333}{3 - e^{0.333} + \cos 0.333}$$

$$\therefore \qquad X_1 = 0.36$$

6.4 NUMERICAL INTEGRATION (QUADRATURE) BY TRAPEZOIDAL AND SIMPSON'S RULES

The general problem of numerical integration may be stated as follows. Given a set of data points (x_0,y_0), (x_1,y_1) ....(x_n,y_n), of a function $y = f(x)$, where $f(x)$ is not known explicitly it is required to compute the value of the definite integral,

$$I = \int_a^b y\,dx \qquad \text{... (i)}$$

As in the case of numerical differentiation, we replace f(x) by an interpolating polynomial $\phi(x)$ and obtain on integration an approximate value of the definite integral. Thus, different integration formulas can be obtained depending upon the type of interpolation formula used.

Let the interval [a,b] be divided into n equal subintervals such that

$$a = x_0 < x_1 < x_2 < \dots\dots\dots < x_n = b.$$

Clearly, $$x_n = x_0 + nh$$

Hence, the integral becomes, $$I = \int_{x_0}^{x_n} y\,dx$$

Approximating y by **Newton's Forward Difference** formula, we obtain,

$$I = \int_{x_0}^{x_n} [y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \dots\dots\dots\dots]dx.$$

Since $x = x_0 + ph$, $dx = hdp$ and hence the above integral becomes

$$h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \dots\right]dp$$

which gives on simplification

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \dots\dots\right]$$

This is known as General formula, we can obtain different integration formulas by putting n = 1, 2, 3 .... etc. We derive here a few of these formulae but it should be remarked that the **Trapezoidal and Simpson's 1/3 rules** are found to give sufficient accuracy for use in practical problems.

The following table shows how Δy_0, Δy_1, $\Delta^2 y_0$ are derived from (x_0, y_0), (x_1, y_1), (x_2, y_2) etc.

x_0	y_0		
		Δy_0	
x_1	y_1		$\Delta^2 y_0$
		Δy_1	
x_2	y_2		

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$

and $$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$$

6.4.1 Trapezoidal Rule

Setting n = 1 in the general formula, all differences higher than the first will become zero and we obtain;

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1] \qquad \text{... (i)}$$

For the next interval $[x_1, x_2]$, we deduce similarly

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}[y_1 + y_2] \qquad \text{... (ii)}$$

and so on. For the last interval $[x_{n-1}, x_n]$, we have

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}[y_{n-1} + y_n] \qquad \text{... (iii)}$$

combining all these expressions, we obtain the rule

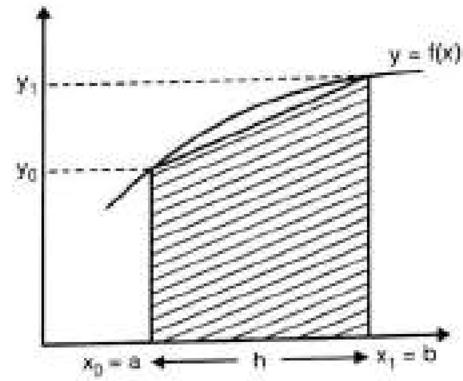
$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$$

which is known as trapezoidal rule.

The geometrical significance of this rule is that the curve $y = f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2) ....; (x_{n-1}, y_{n-1}) and (x_n, y_n).

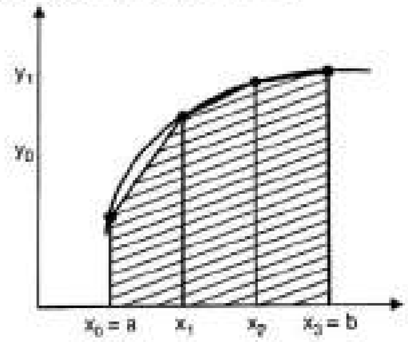
The area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ and $x = x_n$ and the x - axis is then approximately equivalent to the sum of the areas of n **Trapeziums** obtained.

Simple Trapezioidal Rule:



$$\text{Shaded Area} = \text{Area of Trapezium} \cong \int_a^b f(x)dx$$

Compound Trapezoidal Rule (with 4 pts and 3 intervals):



$$\text{Shaded Area} = \text{Sum of Area of 3 trapezium} \cong \int_a^b f(x)dx$$

6.4.2 Simpson's Rules

6.4.2.1 Simpson's 1/3 Rule

This rule is obtained by putting n = 2 in general formula i.e., by replacing the curve by n/2 arcs of second degree polynomials or parabolas. We have been,

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{h}{3}\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly,

$$\int_{x_2}^{x_4} ydx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

and finally

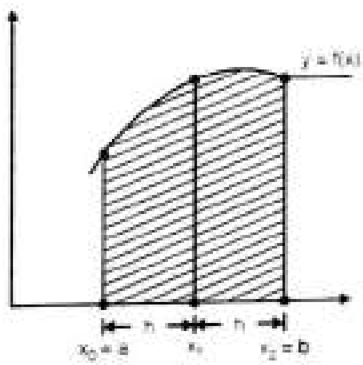
$$\int_{x_{n-2}}^{x_n} ydx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Summing up we obtain,

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2}) + y_n]$$

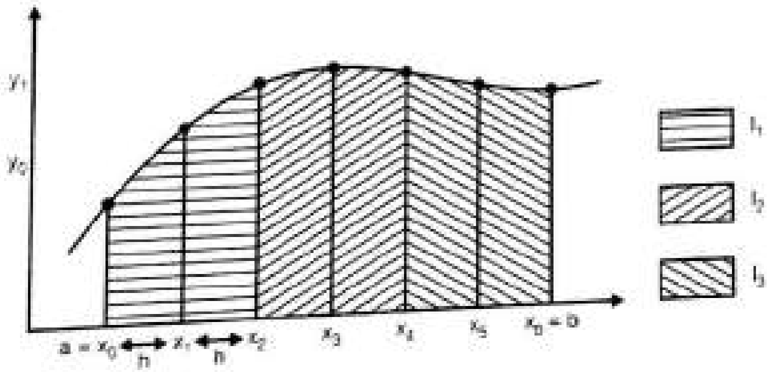
which is known as **"Simpson's 1/3 rule"** or simply **"Simpson's rule"**. It should be noted that this rule requires the divisions of the whole range into an even number of subintervals of width h.

Simple Simpson's Rule:



$$\text{Shaded Area} = \int_a^b f(x)dx$$

Compound Simpson's Rule: (7 pts or 6 intervals)



$$I = \int f(x)dx = I_1 + I_2 + I_3$$

6.4.2.2 Simpson's 3/8 Rule

Setting n = 3 in general formula we observe that all differences higher than the third will become zero and we obtain,

$$\int_{x_0}^{x_3} ydx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, $$\int_{x_3}^{x_6} y dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

and so on. Sumining up all these, we obtain,

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \dots + y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

This rule called **"Simpson's 3/8 rule"**, is not so accurate as Simpson's rule.

ILLUSTRATIVE EXAMPLES

Example:

Evaluate, $I = \int_0^1 \frac{1}{1+x} dx$ correct to three decimal places using (i) Trapezoidal- rule and (ii) Simposon's rules (take h = 0.5) and check which rule is more accurate.

Solution:

We solve this question by both the **Trapezoidal** and **Simpson's** rules with h = 0.5.

The value of x and y are tabulated below.

X	0	0.5	1.0
$y = \frac{1}{1+x}$	1.0000	0.6667	0.5

(a) Trapezoidal rule gives:

$$I = \frac{1}{4}[1.0000 + 2(0.6667) + 0.5] = 0.7084$$

(b) Simpson's rule gives:

$$\frac{1}{6}[1.0000 + 4(0.6667) + 0.5] = 0.6945.$$

Note that the exact answer for this problem by analytical integration method

$$I = \int_0^1 \frac{1}{1+x} dx = [\log_e(1+x)]_0^1 = \log_e 2 = 0.6931$$

Clearly, Simpson's rule is closer to the answer and has less error compared to trapezoidal rule.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.36 The table below gives values of a function F(x) obtained for values of x at intervals of 0.25.

x	0	0.25	0.5	0.75	1.0
F(x)	1	0.9412	0.8	0.64	0.50

The value of the integral of the function between the limits 0 to 1 using Simpson's rule is

(a) 0.7854
(b) 2.3562
(c) 3.1416
(d) 7.5000

[CE, GATE-2010, 2 marks]

Solution: (a)

$$I = \frac{1}{3}h(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4)$$

$$= \frac{1}{3}\times 0.25(1+4\times 0.9412 + 2\times 0.8 + 4\times 0.64 + 0.5) = 0.7854$$

Q.37 The estimate of $\int_{0.5}^{1.5}\frac{dx}{x}$ Obtained using Simpson's rule with three-point function evaluation exceeds the exact value by

(a) 0.235
(b) 0.068
(c) 0.024
(d) 0.012

[CE, GATE-2012, 1 mark]

Solution: (d)

Exact value of $\int_{0.5}^{1.5}\frac{dx}{x} = [\log x]_{0.5}^{1.5}$

$$= \log(1.5) - \log(0.5) = 1.0986$$

Approximate value by Simpson's rule with 3pts is

$$I = \frac{h}{3}(f(0) + 4f(1) + f(2))$$

$$n_i = n_{pt} - 1 = 3 - 1 = 2$$

(n_{pt} is the number of pts and n_i is the number of intervals)

Here $h = \frac{b-a}{n_i} = \frac{1.5-0.5}{2} = 0.5$

The table is

i	x_i	f_i
0	0.5	$\frac{1}{0.5}$
1	1.0	$\frac{1}{1}$
2	1.5	$\frac{1}{1.5}$

$$I = \frac{0.5}{3}\left(\frac{1}{0.5} + 4\times 1 + \frac{1}{1.5}\right) = 1.1111$$

So the estimate exceeds the exact value by

Approximate value – Exact value = 1.1111 – 1.0986 = 0.012499 = 0.012

Q.38 The magnitude of the error (correct to two decimal places) in the estimation of following integral using simpson 1/3 rule. Take the step length as 1

$$\int_0^4 (x^4 + 10)dx$$

[CE, GATE-2013, 2 Mark]

Solution:

x	0	1	2	3	4
y	10	11	26	91	266

Using Simpson's Rule, the estimated value of the integral $\int_0^4 (x^4 + 10)dx$

$$= \frac{1}{3}[(10+266)+2(26)+4(11+91)] = 245.33$$

The exact value of integral

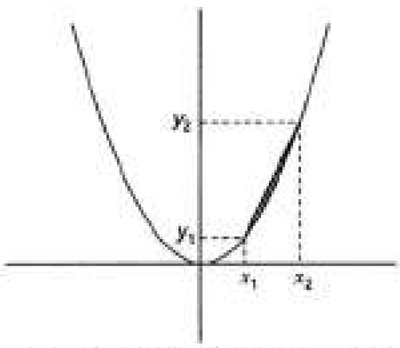
$$\int_0^4 (x^4+10)dx = \left[\frac{x^5}{5}+10x\right]_0^4 = \frac{4^5}{5}+10\times 4 = 244.8$$

$\therefore$ Magnitude of error = |exact value − estimated value| = |244.8 − 245.33| = 0.53

Q.39 The integral $\int_{x_1}^{x_2} x^2 dx$ with $x_2 > x_1 > 0$ is evaluated analytically as well as numerically using a single application of the trapezoidal rule. If I is the exact value of the integral obtained analytically and J is the approximate value obtained using the trapezoidal rule, which of the following statements is correct about their relationship?

(a) $J > I$
(b) $J < I$
(c) $J = I$
(d) Insufficient data to determine the relationship **[CE, GATE-2015 : 1 Mark, Set-I]**

Solution: (a)



Exact value is computed by integration which follows thee exact shape of graph while computing the area.

Whereas, in Trapezoidal rule, the lines joining each points are considered straight lines which is not the exact variation of graph all the time like as shown in figure.

$\therefore$ $J > I$

OR

$$\text{Error} = -\frac{h^3}{12}f''(\xi)\times n_i$$

Here, $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2 > 0$

Since f''(x) is positive, the error is negative.

Since error = exact − approximate.

$= I - J$

and since error is negative in this case J > I is true.

Q.40 For step-size, $\Delta x = 0.4$, the value of following integral using Simpson's 1/3 rule is ________.

$$\int_0^{0.8}\left(0.2+25x-200x^2+675x^3-900x^4+400x^5\right)dx$$

[CE, GATE-2015 : 2 Marks, Set-II]

Solution:

$$a = 0, b = 0.8, \Delta x = 0.4$$
$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

x	0	0.4	0.8
$f(x)$	0.2	2.456	0.232

By Simpson's 1/3 Rule

$$y(x) = \int_0^{0.8}(0.2+25x-200x^2+675x^3-900x^4+400x^5)dx = \frac{4}{3}[y_0+4y_1+y_2]$$

$$y_0 = y(0) = 0.2$$
$$y_1 = y(0.4) = 2.456$$
$$y_2 = y(0.8) = 0.232$$
$$y(n) = \frac{0.4}{3}(0.2+4\times 2.456+0.232) = 1.367$$

Q.41 A calculator has accuracy up to 8 digits after decimal place. The value of $\int_0^{2\pi}\sin x\, dx$ when evaluated using this calculator by trapezoidal method with 8 equal intervals, to 5 significant digits is

(a) 0.00000
(b) 1.0000
(b) 0.00500
(d) 0.00025

[ME, GATE-2007, 2 marks]

Solution: (a)

$$h = \frac{2\pi-0}{8} = \frac{\pi}{4}$$
$$y_0 = \sin(0) = 0$$
$$y_1 = \sin\left(\frac{\pi}{4}\right) = 0.70710$$
$$y_2 = \sin\left(\frac{\pi}{2}\right) = 1$$
$$y_3 = \sin\left(\frac{3\pi}{4}\right) = 0.7010$$
$$y_4 = \sin(\pi) = 0$$
$$y_5 = \sin\left(\frac{5\pi}{4}\right) = -0.70710$$
$$y_6 = \sin\left(\frac{6\pi}{4}\right) = -1$$
$$y_7 = \sin\left(\frac{7\pi}{4}\right) = -0.70710$$
$$y_8 = \sin\left(\frac{8\pi}{4}\right) = 0$$

i	x	y = sin x
0	0	0
1	$\frac{\pi}{4}$	0.70710
2	$\frac{\pi}{2}$	1
3	$3\frac{\pi}{4}$	0.70710
4	π	0
5	$5\frac{\pi}{4}$	−0.70710
6	$6\frac{\pi}{4}$	−1
7	$7\frac{\pi}{4}$	−0.70710
8	2π	0

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} f(x)\cdot dx = \frac{h}{2}[(y_0 + y_n + 2(y_1 + y_2 + ... + y_{n-1})]$$

$$\int_0^{2\pi} \sin x \cdot dx = \frac{\pi}{8} \times [(0+0) + 2(0.70710 + 1 + 0.70710 + 0 - 0.70710 - 0.70710)]$$

$$= 0.00000$$

Q.42 Torque exerted on a flywheel over a cycle is listed in the table. Flywheel energy (in J per unit cycle) using Simpson's rule is

Angle (degree)	0	60	120	180	240	300	360
Torque (N m)	0	1066	–323	0	323	–355	0

(a) 542
(b) 993
(c) 1444
(d) 1986

[ME, GATE-2010, 2 marks]

Solution: (b)

Flywheel energy = $\int_0^{2\pi} T(\theta)d\theta$, where $T(\theta)$ is torque exerted.

The integral by using simpson's rule is

$$I = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6)$$

$$h = 60 \text{ degrees} = \frac{\pi}{3} \text{ radious}$$

$$I = \frac{1}{3} \times \frac{\pi}{3} \times [0 + 4 \times 1066 + 2(-323) + 4(0) + 2(323) + 4(-355) + 0]$$

$$= 993$$

Q.43 The integral $\int_1^3 \frac{1}{x}dx$, when evaluated by using Simpson's 1/3 rule on two equal subintervals each of length 1, equals

(a) 1.000
(b) 1.098
(c) 1.111
(d) 1.120

[ME, GATE-2011, 2 marks]

Solution: (c)

$$I = \int_1^3 \frac{1}{x}dx$$

x	$f(x) = \frac{1}{x}$
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$

$$I = \frac{h}{3}(f_0 + 4f_1 + f_2) = \frac{1}{3}\left(1 + 4 \times \frac{1}{2} \times \frac{1}{3}\right) = 1.111$$

Q.44 The value of $\int_{2.5}^{4} ln(x)\,dx$ calculated using the Trapezoidal rule with five subintervals is ___.

[ME, GATE-2014 : 2 Marks, Set-2]

Solution :

x	2.5	2.8	3.1	3.4	3.7	4
$y = f(x)$	0.1963	1.0296	1.1314	1.2237	1.3083	1.3863
y_x	y_0	y_1	y_2	y_3	y_4	y_5

$$I = \int_{2.5}^{4} ln(x)dx = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$I = \frac{0.3}{2}[(0.9163 + 1.3863) + 2(1.0296 + 1.1314 + 1.2237 + 1.3083)]$$

$$I = \frac{0.3}{2} \times 11.6886 = 1.7533$$

Q.45 The definite integral $\int_1^3 \frac{1}{x}dx$ is evaluated using trapezoidal rule with a step size of 1. The correct answer is ________.

[ME, GATE-2014 : 1 Mark, Set-3]

Solution :

x	1	2	3
$y = f(x)$	1	0.5	0.33
y_n	y_0	y_1	y_2

$$I = \int_1^3 \frac{1}{x}dx = \frac{h}{2}[(y_0 + y_2) + 2y_1]$$

$$= \frac{1}{2}[1 + 0.33 + 2 \times 0.5] = \frac{2.33}{2} = 1.165$$

Q.46 Using a unit step size, the volume of integral $\int_1^2 x\ln x\,dx$ by trapezoidal rule is ________.

[ME, GATE-2015 : 1 Mark, Set-3]

Solution: (0.693)

	y_0	y_n
x	1	2
$f(x)$	0	2(ln2)

$$\therefore \quad I = \frac{h}{2}[y_0 + y_n]$$

$$I = \frac{1}{2}[0 + 2\ln 2] = ln\,2 = 0.693$$

Q.47 Simpson's $\frac{1}{3}$ rule is used to integrate the function $f(x) = \frac{3}{5}x^2 + \frac{9}{5}$ between $x = 0$ and $x = 1$ using the least number of equal sub-intervals. The value of the integral is ______.

[ME, GATE-2015 : 1 Mark, Set-1]

Solution: (2)

$$f(x) = \frac{3}{5}x^2 + \frac{9}{5}$$

x	0	0.5	1
$f(x)$	1.8	1.95	2.4

$$\Rightarrow \quad \int_0^1 f(x) = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$= \frac{0.5}{3}[1.8 + 4(1.95) + 2.4] = 2$$

Q.48 The values of function $f(x)$ at 5 discrete points are given below:

x	0	0.1	0.2	0.3	0.4
$f(x)$	0	10	40	90	160

Using Trapezoidal rule step size of 0.1, the value of $\int_0^{0.4} f(x)dx$ is ______.

[ME, GATE-2015 : 2 Marks, Set-2]

Solution: (22)

$$\int_0^{0.4} f(x)dx = \frac{h}{2}[y_0 + 2[y_1 + y_2 + y_3] + y_4] = \frac{0.1}{2}[0 + 2[10 + 40 + 90] + 160] = 22$$

Q.49 Using the trapezoidal rule, and dividing the interval of integration into three equal subintervals, the definite integral $\int_{-1}^{+1}|x|dx$ is ______.

[ME, GATE-2014 : 2 Marks, Set-2]

Solution :

$$h = \frac{b-a}{n} = \frac{1-(-1)}{3} = \frac{2}{3} = 0.667$$

x	f(x) = |x|
−1	1
−0.333	0.333
+0.333	0.333
1	1

$$I = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + f_3)$$

$$= \frac{0.667}{2}(1 + 2 \times 0.333 + 2 \times 0.333 + 1) = 1.11$$

Q.50 The velocity v (in kilometer/minute) of a motorbike which starts from rest, is given at fixed intervals of time t (in minutes) as follows:

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

The approximate distance (in kilometers) rounded to two places of decimals covered in 20 minutes using Simpson's 1/3rd rule is ______.

[CS, GATE-2015 : 2 Marks, Set-3]

Solution: (309.33)

Given that the motorbike starts from rest.

$\therefore$ At $t = 0$, $v = 0$

So the table now becomes

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

h = Table spacing = 2 minutes

So the distance (in kilometers) covered in 20 minutes using Simpson's rule

$$= \int_0^{20} v\,dt = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + f_{10})$$

$$= \frac{2}{3}(0 + 4\times 10 + 2\times 18 + 4\times 25 + \ldots + 0) = 309.33$$

Q.51 Numerical integration using trapezoidal rule gives the best result for a single variable function, which is

(a) linear
(b) parabolic
(c) logarithmic
(d) hyperbolic

[ME, 2016 : 1 Mark, Set-2]

Solution: (a)

Trapezoidal rule gives the best result in single variable function when the function is linear (degree 1).

Q.52 The error in numerically computing the integral $\int_0^{\pi}(\sin x + \cos x)\,dx$ using the trapezoidal rule with three intevals of equal length between 0 and π is ______.

[ME, 2016 : 2 Marks, Set-2]

Solution:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π
$f(x)$	1 y_0	1.366 y_1	0.366 y_2	−1 y_3

By Trapezoidal

$$\int_0^{\pi}(\sin x + \cos x)dx = \frac{\pi/3}{2}(1+(-1)+2(1.366+0.366))$$

$$= \frac{\pi}{3}(1.732) = 1.1812$$

$$\int_0^{\pi}(\sin x + \cos x)dx = \int_0^{\pi/2}(\sin x + \cos x)dx + \int_{\pi/2}^{\pi}(\sin x + \cos x)dx$$

$$= (-\cos x + \sin x)\Big|_0^{\pi/2} + (-\cos x + \sin x)\Big|_{\pi/2}^{\pi}$$

$$= [(0+1)-(-1+0)] + [(1+0)-(0+1)]$$

$$= 1 + 1 + 1 - 1 = 2$$

Error = Exact value − approx value = 2 − 1.1812 = 0.187

6.4.3 Truncation Error Formulae for Trapezoidal and Simpson's Rule

Let h be the step size used in integration.

The truncation error formula for simple trapezoidal rule with 2 pts is given by

$$T_E = -\frac{h^3}{12} f''(\xi)$$

For composite trapezoidal rule with N_i intervals.

$$T_{E(max)} = -\frac{h^3}{12} N_i f''(\xi)$$

The absolute T_E bound for simple trapezoidal rule is given by

$$|T_E|_{bound} = \max\left|-\frac{h^3}{12} f''(\xi)\right|$$

$$= \frac{h^3}{12}\max|f''(\xi)| \qquad \text{where, } x_0 \le \xi \le x_n$$

For Composite rule also similarly,

$$|T_E|_{bound} = \max\left|-\frac{h^3}{12} N_i f''(\xi)\right|$$

$$= \frac{h^3}{12} N_i \max|f''(\xi)| \qquad \text{where, } x_0 \le \xi \le x_n$$

The truncation error for simple Simpson's rule with 3 pts is given by

$$T_E = -\frac{h^5}{90} f^{iv}(\xi)$$

For composite Simpson's rule with N_i intervals, the truncation error bound is given by

$$T_{E(max)} = -\frac{h^5}{90} f^{iv}(\xi) N_{si}$$

where, N_{si} is number of simpson's intervals.

Since, $N_{si} = \frac{N_i}{2}$

So, $T_{E(max)} = -\frac{h^5}{90}\left(\frac{N}{2}\right) f^{iv}(\xi)$

The absolute truncation error bound for simple Simpson's rule is given by,

$$|T_E|_{bound} = \max\left|-\frac{h^5}{90} f^{iv}(\xi)\right|$$

$$= \frac{h^5}{90} \max |f^{iv}(\xi)| \qquad \text{where, } x_0 \le \xi \le x_n$$

The absolute truncation error bound for composite Simpson's rule with N intervals is given by,

$$|T_E|_{bound} = \max\left|-\frac{h^5}{90}\left(\frac{N}{2}\right) f^{iv}(\xi)\right| = \frac{h^5}{90}\left(\frac{N}{2}\right) \max |f^{iv}(\xi)|$$

$$= \frac{h^5}{180} N_i \max |f^{iv}(\xi)| \qquad \text{where, } x_0 \le \xi \le x_n$$

In all these formulae, $N_i = (b - a)/h$ (where a and b are the limits of integration) and $N_i = N_{pt} - 1$ (where N_{pt} is the number of pts used in the integration). Since T_E for simple trapezoidal rule is proportional to h^2, it is a third order method. i.e. TE = $O(h^2)$. Since T_E for simple simpson's rule is proportional to h^4, it is a fifth order method. i.e. TE = $O(h^4)$.

Important Note:

1. Trapezoidal rule gives exact results while integrating polynomials upto degree = 1.
2. Simpson's rule gives exact results while integrating polynomials upto degree = 3.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.53 A 2nd degree polynomial, f(x) has values of 1, 4 and 15 at x = 0, 1 and 2, respectively. The integral $\int_0^2 f(x)\,dx$ is to be estimated by applying the trapezoidal rule to this data. What is the error (defined as "true value - approximate value") in the estimate?

(a) $-\frac{4}{3}$ (b) $-\frac{2}{3}$

(c) 0 (d) $\frac{2}{3}$ **[CE, GATE-2006, 2 marks]**

Solution: (a)

$$f(x) = 1, 4, 15 \text{ at } x = 0, 1 \text{ and } 2 \text{ respectively}$$

$$\int_0^2 f(x)dx = \frac{h}{2}(f_1 + 2f_2 + f_3) \text{ (3 point Trapezoidal Rule)}$$

here $h = 1$

$\therefore$ $$\int_0^2 f(x)dx = \frac{1}{2}(1 + 2 \times 4 + 15) = 12$$

$\therefore$ Approximate value by Trapezoidal Rule = 12

Since f(x) is second degree polynomial, let

$$f(x) = a_0 + a_1x + a_2x^2$$

$$f(0) = 1$$

$\Rightarrow$ $a_0 + 0 + 0 = 1$

$\Rightarrow$ $a_0 = 1$

$$f(1) = 4$$

$\Rightarrow$ $a_0 + a_1 + a_2 = 4$

$\Rightarrow$ $1 + a_1 + a_2 = 4$

$$\Rightarrow \quad a_1 + a_2 = 3 \quad \ldots(i)$$
$$f(2) = 15$$
$$\Rightarrow \quad a_0 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow \quad 1 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow \quad 2a_1 + 4a_2 = 14 \quad \ldots(ii)$$

Solving (i) and (ii) $a_1 = -1$ and $a_2 = 4$

$$\therefore \quad f(x) = 1 - x + 4x^2$$

Now, exact value of $\int_0^2 f(x)dx = \int_0^2 (1 - x + 4x^2)\,dx = \left[x - \frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$

Error = Exact – Approximate value = $\frac{32}{3} - 12 = -\frac{4}{3}$

Q.54 The accuracy of Simpson's rule quadrature for a step size h is

(a) $O(h^2)$ (b) $O(h^3)$

(c) $O(h^4)$ (d) $O(h^5)$ **[ME, GATE-2003, 1 mark]**

Solution: (c)

Q.55 The minimum number of equal length subintervals needed to approximate $\int_1^2 xe^x\,dx$ to an accuracy of at least $1/3 \times 10^{-6}$ using the trapezoidal rule is

(a) 1000e (b) 1000

(c) 100e (d) 100 **[CS, GATE-2008, 2 marks]**

Solution: (a)

Here, the function being integrated is

$$f(x) = xe^x$$
$$f'(x) = xe^x + e^x = e^x(x+1)$$
$$f''(x) = xe^x + e^x + e^x = e^x(x+2)$$

Since, both e^x and x are increasing functions of x, maximum value of $f''(\xi)$ in interval $1 \le \xi \le 2$, occurs at $\xi = 2$.

So, $\max|f''(\xi)| = e^2(2+2) = 4e^2$

Truncation Error for trapezoidal rule = TE (bound)

$$= \frac{h^3}{12}\max|f''(\xi)| \cdot N_i$$

where N_i is number of subintervals

$$N_i = \frac{b-a}{h}$$

$$\therefore \quad T_{e(bound)} = \frac{h^3}{12}\max|f''(\xi)| \cdot \frac{b-a}{h}$$
$$= \frac{h^2}{12}(b-a)\max|f''(\xi)| \quad 1 \le \xi \le 2$$
$$= \frac{h^2}{12}(2-1)(4e^2) = \frac{h^2}{3}e^2$$

Now putting $T_{e(bound)} = \frac{1}{3} \times 10^{-6}$

We get $\frac{h^2}{3}e^2 = \frac{1}{3} \times 10^{-6}$

$\Rightarrow \quad h^2 = \frac{10^{-6}}{e^2}$

$\Rightarrow \quad h = \frac{10^{-3}}{e}$

Now, Number of Intervals $= N_i = \frac{b-a}{h} = \frac{2-1}{(10^{-3}/e)} = 1000\,e$

Q.56 With respect to the numerical evaluation of the definite integral $K = \int_a^b x^2 dx$, where a and b are given, which of the following statements is/are TRUE?

(I) The value of K obtained using the trapezoidal rule is always greater than or equal to the exact value of the definite integral.

(II) The value of K obtained using the Simpson's rule is always equal to the exact value of the definite integral

(a) I only
(b) II only
(b) Both I and II
(d) Neither I nor II

[CS, GATE-2014 : 2 Marks, Set-3]

Solution : (c)

While computing $K = \int_a^b x^2 dx$

Error = Exact value – Approximate value

For trapezoidal rule

Error $= -\frac{h^3}{12} f''(\xi) \times n_i$

Since h and n_i are always positive, sign of the error is controled only by the sign of $f''(\xi)$.

Here $f(x) = x^2$ so $f''(x) = 2$ which is always positive. So the sign of the error is always negative.

i.e. approximate value always greater than or equal to the exact value of the integral.

So (I) is true.

Similarly for **Simpson's rule**

Error $= -\frac{h^5}{90} f''''(\xi) \times n_i$

Since h and n_i are always positive, sign of the error is controled only by the sign of $f''''(\xi)$.

Here $f(x) = x^2$ so $f''''(x) = 0$.

So the error is always 0. i.e. approximate value always equal to the exact value of the integral.

So (II) is true.

Therefore both (I) and (II) are correct.

6.5 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

6.5.1 Introduction

Analytical methods of solution are applicable only to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now available which reduce the time taken to do numerical computation considerably.

A number of numerical methods are available for the solution of first order differential equations of the form:

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0. \quad \text{... (i)}$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The method of Picard and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are therefore, termed as step-by-step methods.

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams-Bashforth method may be-applied for finding y over a wider range of x-values. These later methods require starting values which are found by Picard's or Taylor series or Runge-Kutta methods.

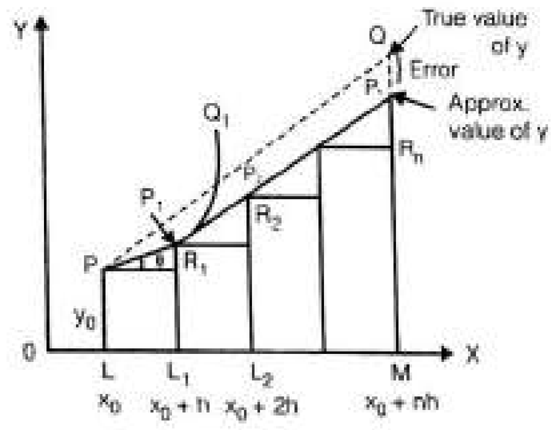
The initial condition in (i) is specified at the point x_0. Such problems in which all the initial conditions are given at the initial point only are called initial value problems. But there are problems where conditions are given at two or more points. These are known as boundary value problems. In this chapter, we shall study three methods common used for solution of first order differential equations, namely.

1. Euler's Method
2. Modified Euler's Method
3. Runge-Kutta Method of Fourth Order (Classical Runge-Kutta Method)

6.5.2 Euler's Method

Consider the equation, $dy/dx = f(x, y)$ ... (i)

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown in Fig. Now we have to find the ordinate of any other point Q on this curve.



Let us divide LM into n sub-intervals each of width h at L_1, L_2, ... so that h is quiet small. In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y = L_1P_1 = LP + R_1P_1$$

$$= y_0 + PR1 \tan\theta = y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + h\, f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (i) through P_1 and let its tangent at P_1 meet the ordinate through L2 in $P_2(x_0 + 2h, y_2)$. Then

Repeating this process n times, we finally reach an approximation MP_n of MQ given by

$$y_{n+1} + f(x_0 + (n-1)h, y_{n-1})$$

in general we may write

$$y_{i+1} = y_i + h\,f(x_i, y_i)$$

This is Euler's method of finding an approximate solution of (i).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, i.e. by a sequence of short lines. Unless h is small, the error is bound to be quiet significant. This sequence of lines may also deviate considerably from the curve of solution. Hence there is a modification of this method which is given in the next section, called modified Euler's method, which is more accurate.

ILLUSTRATIVE EXAMPLES

Example:

Using Euler's method, find an approximate value of y corresponding to x = 1, given that dy/dx = x + y and y = 1 when x = 0.

Solution:

We take n = 10 and h = 0.1 which is sufficiently small. The various calculations are arranged as follows:

x	y	x + y = dy/dx	old y + 0.1(dy/dx) = new y
0.0	1.00	1.00	1.00 + 0.1(1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1(1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1(1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1(1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1(1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1(2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1(2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1(2.89) = 2.48
0.8	2.48	3.89	2.48 + 0.1(3.89) = 2.81
0.9	2.81	3.71	2.81 + 0.1(3.71) = 3.1
1.0	3.18		

Thus the required approximate value of y is 3.18 at x = 1.0.

Obs. In this example, the true value of y from its exact solution at x = 1 is

$$y = 2e^x - x - 1$$

$$2e^1 - 1 - 1 = 3.44$$

whereas by Euler's method y = 3.18. In the above solution, had we chosen n = 20, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple, but cannot be considered as one of the best.

Example:

Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition y = 1 at x = 0; find y for x = 0.1 by Euler's method.

Solution:

We divide the interval (0, 0.1) into five steps i.e. we take n = 5, $h = \frac{b-a}{n} = \frac{0.1-0}{5} = 0.02$.

The various calculations are arranged as follows:

x	y	x + y = dy / dx	old y + h(dy / dx) = new y
0.00	1.0000	1.0000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.200 + 0.02(9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02(926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02(893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02(802) = 1.0928
0.10	1.0928		

Hence the required approximate value of y = 1.0928.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.57 The ordinary differential equation

$$\frac{dx}{dt} = -3x + 2, \text{ with } x(0) = 1$$

is to be solved using the forward Euler method. The largest time step that can be used to solve the equation without making the numerical solution unstable is ________.

[EC, 2016 : 2 Marks, Set-2]

Solution:

$$\frac{dy}{dx} = -3y + 2, \qquad y(0) = 1$$

If $|1 - 3h| < 1$, then solution of differential equation is stable.

$$-1 < 1 - 3h < 1$$
$$-2 < -3h < 0$$
$$0 < h < \frac{2}{3}$$
$$h_{max} = \frac{2}{3} = 0.66$$

6.5.3 Modified Euler's Method

In Euler's method $\quad y_{i+1} = y_i + h\, f(x_i, y_i)$

In Backward Euler's method $\quad y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1}) \quad$ ... (i)

A numerical method where y_{i+1} appears on LHS and RHS of the iterative equation is called an implicit method. So Backward Euler's method is an Implicit method, while Euler's method is explicit since y_{i+1} appears only on left side of iterative equation.

In Backward Euler's method, we need to rearrange and solve (i) for y_{i+1} before proceeding further.

ILLUSTRATIVE EXAMPLES

Example:

Using Backward Euler's Method find an approximate value of y corresponding to x = 0.2, given that dy/dx = x + y and y = 1 when x = 0, use step size h = 0.1.

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$
$$y_{i+1} = y_i + h(x_{i+1} + y_{i+1})$$

Solution:

Solving for y_{i+1} we get, $y_{i+1} = \dfrac{y_i + h\,x_{i+1}}{1-h}$

Now the calculations are shown below:

i	x_i	y_i	Comments
0	0.0	1.00	Initial condition given
1	0.1	1.122	$y_1 = \dfrac{y_0 + h\,x_1}{1-h} = \dfrac{1+0.1\times 0.1}{1-0.1} = 1.122$
2	0.2	1.2689	$y_2 = \dfrac{y_1 + h\,x_2}{1-h} = \dfrac{1.122+0.1\times 0.2}{1-0.1} = 1.2689$

So, the approximate value of y at x = 0.2 is 1.2689.

Notice that this same problem when solved by forward Euler's method, gave a slightly different answer for y which was y = 1.22 at x = 0.2.

The advantage of Backward Euler's method is its stability. Backward Euler's method is more stable compared to forward Euler's method.

A method is stable if the effect of any single fixed round off error is bounded, independent of the number of mesh points.

6.5.4 Runge-Kutta Method

The Taylor's series method of solving differential equations numerically is restricted by the labour
involved in finding the higher order derivatives. However there is a class of methods known as
Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods
agree with Taylor's series solution upto the terms in h^r, where r differs from method to method and is
called the order of that method. Euler's method Modified Euler's method and Runge's method are the
Runge-Kutta methods of the first, second and third order respectively.
The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-
Kutta' method' or classical Runge-Kutta method.

Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \text{ is as follows:}$$

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = h\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = h\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4).

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

ILLUSTRATIVE EXAMPLES

Example:

Apply Runge-Kutta fourth order method to find an approximate value of y when x = 0.2 given that dy/dx = x + y and y = 1 when x = 0.

Solution:

Here, $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$

$$= \frac{1}{6} \times (1.4568) = 0.2428$$

Now,

$$y_1 = y_0 + k = 1 + 0.2428 = 1.2428$$

Hence the required approximate value of y is 1.2428.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.58 Consider the first order initial value problem

$y' = y + 2x - x^2$, $y(0) = 1$, $(0 \le x < \infty$

with exact solution $y(x) = x^2 + e^x$. For $x = 0.1$, the percentage difference between the exact solution and the solution obtained using a single iteration of the second-order Runge-Kutta method with step-size $h = 0.1$ is ___.

[EC, 2016 : 1 Mark, Set-3]

Solution:

$$\frac{dy}{dx} = y + 2x - x^2$$

$$y(0) = 1$$

$0 \le x \le \infty$

$$f(x, y) = y + 2x - x^2$$

$x_0 = 0$; $y_0 = 1$; $h = 0.1$

$$k_1 = hf(x_0, y_0) = 0.1(1 + 2 \times 0 - 0^2) = 0.1$$

$k_2 = hf(x_0 + h, y_0 + k_1)$
$= 0.1((y_0 + k_1) + 2(x_0 + h) - (x_0 + h)^2)$
$= 0.1((1 + 0.1) + 2(0.1) - (0.1)^2)$
$= 0.129$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.1 + 0.129) = 1.1145$$

Exact solution $y(x) = x^2 + e^x = (0.1)^2 + e^{0.1} = 1.1152$

Error $= 1.1152 - 1.1145 = 0.00069$

% error $= 0.06\%$

6.5.5 Stability Analysis

If the effect of round off error remains bounded as $j \to \infty$, with a fixed step size, then the method is said to be stable; otherwise unstable. Unstable methods will diverge away from solution and cause overflow error.

Using a general single step method equation ... (i)

$$y_{j+1} = E \cdot y$$

Condition for absolute stability is

$$|E| \le 1$$

Using a test equation $y' = \lambda y$,

let us find the condition for stability for Euler's method.

Euler's method equation is

$$y_{j+1} = y_j + h\,f(x_j, y_j)$$
$$= y_j + h\lambda y_j$$
$$= (1 + h\lambda)y_j$$

Now, comparing with (i) we get

$$E = 1 + h\lambda$$

Condition for stability if $|E| < 1$

$$|1 + h\lambda| < 1$$
$$-1 < 1 + h\lambda < 1$$

So, condition for stability is

$$-2 < \lambda h < 0$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.59 The differential equation $(dy/dx) = 0.25y^2$ is to be solved using the backward (implicit) Euler's method with the boundary condition $y = 1$ at $x = 0$ and with a step size of 1. What would be the value of y at $x = 1$?

(a) 1.33
(b) 1.67
(c) 2.00
(d) 2.33

[CE, GATE-2006, 1 mark]

Solution: (c)

$$\frac{dy}{dx} = 0.25y^2 \quad (y = 1 \text{ at } x = 0)$$
$$h = 1$$

Iterative equation for backward (implicit) Euler methods for above equation would be

$$y_{k+1} = y_k + h\,f(x_{k+1}, y_{k+1})$$
$$y_{k+1} = y_k + h \times 0.25\,y_{k+1}^2$$

$$\Rightarrow \quad 0.25h\,y_{k+1}^2 - y_{k+1} + y_k = 0$$

putting $k = 0$ in above equation

$$0.25h\,y_1^2 - y_1 + y_0 = 0$$

since, $y_0 = 1$ and $h = 1$

$$0.25\,y_1^2 - y_1 + 1 = 0$$

$$\Rightarrow \quad y_1 = \frac{1 \pm \sqrt{1-1}}{2 \times 0.25} = 2$$

$$\Rightarrow \quad y_1 = 2$$

Q.60 The error in $\left.\frac{d}{dx}f(x)\right|_{x=x_0}$ for a continuous function estimated with $h = 0.03$ using the central difference formula $\left.\frac{d}{dx}f(x)\right|_{x=x_0} = \frac{f(x_0+h)-f(x_0-h)}{2h}$, is 2×10^{-3}. The values of x_0 and $f(x_0)$ are 19.78 and 500.01, respectively. The corresponding error in the central difference estimate for $h = 0.02$ is approximately

(a) 1.3×10^{-4}
(b) 3.0×10^{-4}
(c) 4.5×10^{-4}
(d) 9.0×10^{-4}

[CE, GATE-2012, 2 marks]

Solution: (d)

Error in central difference formula is $0(h^2)$

This means,

$$\text{error} \propto h^2$$

If error for $h = 0.03$ is 2×10^{-3}

then

Error for $h = 0.02$ is approximately

$$2 \times 10^{-3} \times \frac{(0.02)^2}{(0.03)^2} \simeq 9 \times 10^{-4}$$

Q.61 Match the correct pairs

Numerical Integration Scheme		Order of Fitting Polynomial
P. Simpson's 3/8 Rule	1.	First
Q. Trapezoidal Rule	2.	Second
R. Simpson's 1/3 Rule	3.	Third

(a) P-2, Q-1, R-3
(b) P-3, Q-2, R-1
(c) P-1, Q-2, R-3
(d) P-3, Q-1, R-2

[ME, GATE-2013, 1 Mark]

Answer: (d)

Q.62 Consider an ordinary differential equation $\frac{dx}{dt} = 4t + 4$. If $x = x_0$ at $t = 0$, the increment in x calculated using Runge-Kutta fourth order multi-step method with a step size of $\Delta t = 0.2$ is

(a) 0.22
(b) 0.44
(c) 0.66
(d) 0.88

[ME, GATE-2014 : 2 Marks, Set-2]

Solution : (d)

$$\frac{dx}{dt} = 4t + 4 = f(t_0, x_0)$$

At t = 0, $x = x_0$

Irrespective of values of x, $f(t_0, x_0)$ depends on t only.

$$k_1 = h\, f(t_0, x_0) = 0.2x \times 4 = 0.8$$

$$k_2 = hf\left(t_0 + \frac{1}{2}h, x_0 + \frac{k_1}{2}\right) = hf(0.1, x_0 + 0.4) = 0.2(4 \times 0.1 + 4) = 0.88$$

$$k_3 = hf\left(t_0 + \frac{1}{2}h, x_0 + \frac{k_2}{2}\right) = 0.2f(0.1, x_0 + 0.44) = 0.2(4 \times 0.1 + 4) = 0.88$$

$$k_4 = hf(t_0 + h, x_0 + k_3) = 0.2f(0.2, x_0 + 0.88) = 0.2(4 \times 0.2 + 4) = 0.96$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.8 + 2 \times 0.88 + 2 \times 0.88 + 0.96) = 0.88$$

Q.63 While numerically solving the differential equation $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 1$ using Euler's predictor-corrector (improved Euler-Cauchy) with a step size of 0.2, the value of y after the first step is

(a) 1.00 (b) 1.03

(c) 0.97 (d) 0.96

[IN, GATE-2013 : 2 marks]

Solution: (d)

$$\frac{dy}{dx} + 2xy^2 = 0$$

$$\therefore \quad \frac{dy}{dx} = -2xy^2$$

after one iteration

$$y_1^* = y_0 + h[-2x_0 y_0^2] = 1 + 0.2[-2 \times 0 \times 1^2] = 1 + 0 = 1$$

$$y_1 = y_0 + \frac{1}{2} \times (0.2)[-2x_0 y_0^2 - 2x_1 y_1^*]$$

$$= 1 + 0.1[-(2 \times 0 \times 1^2) - (2 \times 0.2 \times 1)]$$

$$= 1 + 0.1[-0 - 0.4] = 1 - 0.04 = 0.96$$

Q.64 The differential equation $(dx/dt) = [(1 - x)/\tau]$ is discretised using Euler's numerical integration method with a time step $\Delta T > 0$. What is the maximum permissible value of ΔT to ensure stability of the solution of the corresponding discrete time equation?

(a) 1 (b) $\tau/2$

(c) τ (d) 2τ

[EE, GATE-2007, 2 marks]

Solution: (d)

Here, $$\frac{dx}{dt} = \frac{1-x}{\tau}$$

Here, $$f(x, y) = \frac{1-x}{\tau}$$

Euler's Method Equation is

$$x_{j+1} = x_j + h\, f(x_j, y_j)$$

$$\Rightarrow \quad x_{j+1} = x_j + h\left(\frac{1-x_j}{\tau}\right)$$

$$\Rightarrow \quad x_{j+1} = \left(1-\frac{h}{\tau}\right)x_j + \frac{h}{\tau}$$

For stability $\left|1-\frac{h}{\tau}\right| < 1$

$$\Rightarrow \quad -1 \le 1 - \frac{h}{\tau} \le 1$$

Since, $h = \Delta T$ here,

$$-1 \le 1 - \frac{\Delta T}{\tau} < 1$$

$$\Rightarrow \quad \Delta T < 2\tau$$

So, maximum permissible value of ΔT is 2τ.

Q.65 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I

A. Newton-Raphson method
B. Rung-kutta method equations
C. Simpson's Rule equations
D. Gauss elimination

List-II

1. Solving nonlinear equations
2. Solving simultaneous linear equations
3. Solving ordinary differential
4. Numerical integration
5. Interpolation
6. Calculation of Eigenvalues

Codes:

	A	B	C	D
(a)	6	1	5	3
(b)	1	6	4	3
(c)	1	3	4	2
(d)	5	3	4	1

[EC, GATE-2005, 2 marks]

Solution: (c)

Q.66 Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with the initial condition $y(0) = 0$. Using Euler's first order method with a step size of 0.1, the value of $y(0.3)$ is

(a) 0.01
(b) 0.031
(c) 0.0631
(d) 0.1

[EC, GATE-2010, 2 marks]

Solution: (b)

$$\frac{dy}{dx} - y = x, \quad y(0) = 0$$

$$\text{step size} = h = 0.1$$

Euler's first order formula is

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

Here,

$$x_0 = 0, y_0 = y(x_0) = y(0) = 0$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$f(x, y) = \frac{dy}{dx} = y + x$$

$\Rightarrow$

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 0 + 0.1 \times f(0, 0) \\ &= 0 + 0.1 \times (0 + 0) \\ &= 0 \end{aligned}$$

Now,

$$x_1 = 0.1, y_1 = 0$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.1 = 0.2$$

$\Rightarrow$

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) \\ &= 0 + 0.1 \times f(0.1, 0) = 0 + 0.1(0.1 + 0) = 0.01 \end{aligned}$$

Now,

$$x_2 = 0.2, y_2 = 0.01$$

$$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3$$

$\Rightarrow$

$$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) \\ &= 0.01 + 0.1 \times f(0.2, 0.01) = 0.01 + 0.1(0.2 + 0.01) = 0.031 \end{aligned}$$

$\therefore$ at $x_3 = 0.3$, $y_3 = 0.031$.

$\therefore$ Correct answer is choice (b).

OOOO

CHAPTER 7

Laplace Transforms

7.1 INTRODUCTION

The Laplace transform method solve differential equations and corresponding initial and boundary value problems. The process of solution consists of three main steps:

Ist step. The given "hard" problem is transformed into a "simple" equation **(subsidiary equation).**

2nd step. The subsidiary equation is solved by purely algebraic manipulations.

3rd step. The solution of the subsidiary equation is transformed back to obtain the solution of the given problem.

In this way Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. This process is made easier by tables of functions and their transforms, whose role is similar to that of integral tables in calculus.

This switching from operations of calculus to algebraic operations on transforms is called **operational calculus,** a very important area of applied mathematics, and for the engineer, the Laplace transform method is practically the most important operation method. It is particularly useful in problems where the mechanical or electrical driving method. It is particularly useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function, not merely a sine or cosine. Another operational method is the Fourier transform.

The Laplace transform also has the advantage that it solve initial value problems directly, without first determining a general solution. It also solves nonhomogeneous differential equations directly without first solving the corresponding homogeneous equation.

System of ODES and partial differential equations can also be treated by Laplace transforms.

7.2 DEFINITION

Let f(t) be a function of t defined for all positive values of t. Then the Laplace transforms of f(t), denoted by L{f(t)} is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad \ldots (i)$$

provided that the integral exists, s is a parameter which may be a real or complex number.

L{f(t)} being clearly a function of s is briefly written as $\bar{f}(s)$ or as F(s).

i.e. $$L\{f(t)\} = \bar{f}(s),$$

which can also be written as

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

Then f(t) is called the inverse Laplace transform of $\bar{f}(s)$. The symbol L. Which transforms f(t) into $\bar{f}(s)$, is called the Laplace transformation operator.

Example:

If
$$f(t) = 1$$
$$L[f(t)] = \int e^{-st} \cdot 1\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{e^{-\infty} - e^{0}}{-s} = \frac{1}{s}$$

Similarly Laplace transforms of other common functions can also be evaluated and is shown below:

7.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae:

1. $L(1) = \dfrac{1}{s}$ $(s > 0)$
2. $L(t^n) = \dfrac{n!}{s^{n+1}}$, when n = 0, 1, 2, 3,.... $\left[\text{otherwise } \dfrac{\Gamma(n+1)}{s^{n+1}}\right]$
3. $L(e^{at}) = \dfrac{1}{s-a}$ $(s > a)$
4. $L(\sin at) = \dfrac{a}{s^2+a^2}$ $(s < 0)$
5. $L(\cos at) = \dfrac{s}{s^2+a^2}$ $(s > 0)$
6. $L(\sinh at) = \dfrac{a}{s^2-a^2}$ $(s > |a|)$
7. $L(\cosh at) = \dfrac{s}{s^2-a^2}$ $(s > |a|)$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 If L defines the Laplace Transform of a function, L [sin (at)] will be equal to

(a) $\dfrac{a}{s^2-a^2}$ (b) $\dfrac{a}{s^2+a^2}$

(c) $\dfrac{s}{s^2+a^2}$ (d) $\dfrac{s}{s^2-a^2}$

[CE, GATE-2003, 2 marks]

Answer: (b)

Q.2 Laplace transform for the function f(x) = cosh(ax) is

(a) $\dfrac{a}{s^2-a^2}$ (b) $\dfrac{s}{s^2-a^2}$

(c) $\dfrac{a}{s^2+a^2}$ (d) $\dfrac{s}{s^2+a^2}$ **[CE, GATE-2009, 2 marks]**

Solution: (b)

It is a standard result that

$$L(\cosh at) = \frac{s}{s^2-a^2}$$

Q.3 Laplace transform of the function $\sin \omega t$ is

(a) $\frac{s}{s^2+\omega^2}$ (b) $\frac{\omega}{s^2+\omega^2}$

(c) $\frac{s}{s^2-\omega^2}$ (d) $\frac{\omega}{s^2-\omega^2}$

[ME, GATE-2003, 2 marks]

Solution: (b)

$$L[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$$

Q.4 The function f(t) satisfies the differential equation $\frac{d^2f}{dt^2}+f=0$ and the auxiliary conditions, $f(0)=0,\ \frac{df}{dt}(0)=4$. The Laplace transform of f(t) is given by

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$

(c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^2+1}$

[ME, GATE-2013, 2 Marks]

Solution: (c)

$$L\left\{\frac{d^2f}{dt^2}+f\right\} = 0$$

$$L[f] = F[s]$$

$$L[f''] = s^2F(s) - sf(0) - f'(s) = s^2F(s) - 4$$

$$s^2F(s) - 4 + F(s) = 0$$

$$(s^2+1)F(s) = 4$$

$$F(s) = \frac{4}{s^2+1}$$

$$L[f] = \frac{4}{s^2+1}$$

Q.5 If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform $F(s)$ is defined as

(a) $\int_0^{\infty} e^{st} f(t)\,dt$ (b) $\int_0^{\infty} e^{-st} f(t)\,dt$

(c) $\int_0^{\infty} e^{ist} f(t)\,dt$ (d) $\int_0^{\infty} e^{-ist} f(t)\,dt$

[ME, 2016 : 1 Mark, Set-1]

Solution: (b)

$$L(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt$$

Q.6 Consider the function $f(x) = 2x^3 - 3x^2$ in the domain [−1, 2]. The global minimum of $f(x)$ is____.

[ME, 2016 : 2 Marks, Set-1]

Solution:

$$f(x) = 2x^3 - 3x^2 \text{ in } [-1, 2]$$
$$f'(x) = 6x^2 - 6x$$
$$f'(x) = 0$$
$$6x^2 - 6x = 0$$
$$6x(x-1) = 0$$
$$x = 0, 1$$
$$f''(x) = 12x - 6$$
$$f''(0) = -6 \quad \text{Max}$$
$$f''(1) = 6 \quad \text{Min}$$

$x = -1$ $f(-1) = -5$ G. Min.
$x = 2$ $f(2) = 4$
$x = 0$ $f(0) = 0$
$x = 1$ $f(1) = -1$

G. Minima is −5 at $x = 1$.

Q.7 Laplace transform of cos (ωt) is

(a) $\frac{s}{s^2+\omega^2}$

(b) $\frac{\omega}{s^2+\omega^2}$

(c) $\frac{s}{s^2-\omega^2}$

(d) $\frac{\omega}{s^2-\omega^2}$

[ME, 2016 : 1 Mark, Set-2]

Solution: (a)

$$L(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

Q.8 Solutions of Laplace's equation having continuous second-order partial derivatives are called

(a) biharmonic functions
(b) harmonic functions
(c) conjugate harmonic functions
(d) error functions

[ME, 2016 : 1 Mark, Set-3]

Solution: (b)

Solution of laplace equation having continuous
Second order partial derivating

$\therefore$ $$\nabla^2\phi = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\therefore$ ϕ is harmonic function.

7.4 PROPERTIES OF LAPLACE TRANSFORMS

7.4.1 Linearity Property

If a, b, c be any constants and f, g, h any functions of t, then

L[af(t) + bg(t) – ch(t)] = a L{f(t)} + bL[g(t)] – cL{h(t)}

7.4.2 First Shifting Property

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{e^{at}f(t)\} = \bar{f}(s-a)$$

Application of this propety leads us to the following useful results:

1. $L(e^{at}) = \dfrac{1}{s-a}$ $\quad \left[\because L(1) = \dfrac{1}{s}\right]$
2. $L(e^{at}\, t^n) = \dfrac{n!}{(s-a)^{n+1}}$ (n is positive integer) $\quad \left[\because L(t^n) = \dfrac{n!}{s^{n+1}}\right]$
3. $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 + b^2}$ $\quad \left[\because L(\sin bt) = \dfrac{b}{s^2+b^2}\right]$
4. $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$ $\quad \left[\because L(\cos bt) = \dfrac{s}{s^2+b^2}\right]$
5. $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2 - b^2}$ $\quad \left[\because L(\sinh bt) = \dfrac{b}{s^2-b^2}\right]$
6. $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}$ $\quad \left[\because L(\cosh bt) = \dfrac{s}{s^2-b^2}\right]$

where in each case $s > a$.

7.4.3 Change of Scale Property

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$

Proof:

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$$

$$= \int_0^\infty e^{-su/a} f(u).du/a \qquad \left|\begin{array}{l}\text{Put } at = u \\ \Rightarrow dt = du/a\end{array}\right.$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}\bar{f}(s/a).$$

7.4.4 Existence Conditions

$\int_0^\infty e^{-st} f(t)dt$ exists if $\int_0^\lambda e^{-st} f(t)dt$ can actually be evaluated and its limit as $\lambda \to \infty$ exists. Otherwise we may use the following theorem:

If f(t) is continuous and $\underset{t\to\infty}{Lt}\,[e^{-at}f(t)]$ is finite; then the Laplace transform of f(t), i.e. $\int_0^\infty e^{-st} f(t)dt$ exists for $s > a$.

It should however, be noted that the above conditions are sufficient rather than necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$. Similarly a function f(t) for which $\underset{t\to\infty}{Lt}\,[e^{-at}f(t)]$ is finite and having a finite discontinuity will have a Laplace transform for $s > a$.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.9 Laplace transform of cos(ωt) is $\frac{s}{s^2+\omega^2}$. The laplace transform of $e^{-2t}\cos(4t)$ is

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$

(c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (d)

$$L(e^{at}\cos bt) = \frac{s+2}{(s+2)^2+b^2}$$

$$a = -2, b = 4$$

$$\therefore \quad L[e^{-2t}\cos(4t)] = \frac{s+a}{(s+a)^2+16}$$

Q.10 The Laplace transform of e^{i5t} where $i = \sqrt{-1}$, is

(a) $\frac{s-5i}{s^2-25}$ (b) $\frac{s+5i}{s^2+25}$

(c) $\frac{s+5i}{s^2-25}$ (d) $\frac{s-5i}{s^2+25}$

[ME, GATE-2015 : 1 Mark, Set-2]

Solution: (b)

$$e^{i5t} = \cos 5t + i\sin 5t$$

$$L[e^{(i5t)}] = \frac{s}{s^2+25} + \frac{5i}{s^2+25} = \frac{s+5i}{s^2+25}$$

Q.11 Let $X(s) = \frac{3s+5}{s^2+10s+21}$ be the Laplace Transform of a signal x(t). Then, $x(0^+)$ is

(a) 0 (b) 3

(c) 5 (d) 21 **[EE, GATE-2014 : 1 Mark, Set-1]**

Solution : (b)

Given, $$X(s) = \left[\frac{3s+5}{s^2+10s+21}\right]$$

Using initial value theorem,

$$x(0^+) = \lim_{s\to\infty}[sX(s)]$$

$$\therefore \quad x(0^+) = \lim_{s\to\infty}\left[\frac{s(3s+5)}{s^2+10s+21}\right] = \lim_{s\to\infty}\left[\frac{3+\frac{5}{s}}{1+\frac{10}{s}+\frac{21}{s^2}}\right] = \frac{3}{1} = 3$$

Q.12 With initial values $y(0) = y'(0) = 1$, the solution of the differential equation

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ at x = 1 is ____. **[EC, GATE-2014 : 2 Marks, Set-4]**

Solution :

Given

$$y(0) = y'(0) = 1$$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4y = 0 \quad \text{...(i)}$$

Taking the Laplace transform of equation (i), we get

$s^2Y(s) - sY(0) - y'(0) + 4[sY(s) - y(0)] + 4Y(s) = 0$

$$[s^2 + 4s + 4]Y(s) = sY(0)\, y'(0) + 4y(0)$$

$$[s2 + 4s + 4]\, Y(s) = s + 1 + 4$$

$$Y(s) = \frac{s+5}{(s^2+4s+4)} = \frac{(s+5)}{(s+2)^2} = \frac{1}{(s+2)} + \frac{3}{(s+2)^2}$$

$$y(x) = e^{-2x} + 3x\, e^{-2x}$$

at $x = 1$, $\quad y(x) = e^{-1} + 3e^{-2} = 0.77$

Q.13 The Laplace Transform of $f(t) = e^{2t}\sin(5t)\,u(t)$ is

(a) $\frac{5}{s^2 - 4s + 29}$ (b) $\frac{5}{s^2 + 5}$

(c) $\frac{s-2}{s^2 - 4s + 29}$ (d) $\frac{5}{s+5}$ **[EE, 2016 : 1 Mark, Set-1]**

Solution: (a)

Laplace transform of $\sin 5t\ u(t) \rightarrow \frac{5}{s^2 + 25}$

$$e^{2t}\sin 5t\ u(t) \rightarrow \frac{5}{(s-2)^2 + 25} = \frac{5}{s^2 - 4s + 29}$$

7.4.5 Transforms of Derivatives

1. If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

2. If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then

$$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0)$$

7.4.5.1 Differential Equations, Initial Value Problems

We shall now discuss how the Laplace transform method solved differential equations.

We begin with an initial value problem.

$$y'' + ay' + by = r(t). \quad \text{...(i)}$$

$$y(0) = K_0,\ y'(0) = K_1$$

with constant a and b. Here (r) is the **input** (driving force) applied to the mechanical system and y(t) is the **output** (response of the system). In Laplace's method we do three steps.

1st Step: Taking Laplace transform of LHS and RHS of 1 we get

$$L(y'') + a\,L(y') + b\,L(y) = L(r).$$

Now substituting $L(y') = sL(y) - f(0)$ and $\boxed{L(y'') = s^2 L(y) - sf(0) - f'(0).}$, we get

$[s^2L(y) - sy(0) - y'(0)] + a[sL(y) - y(0)] + by = L(r).$

Now writing $Y = L(y)$ and $R = L(r)$. This gives

$[s^2Y(s) - sy(0) - y'(0)] + a[sY(s) - y(0)] + bY = R(s)$

This is called the subsidiary equation. Collecting Y-terms, we have

$$(s^2 + as + b)Y(s) = (s + a)\,y(0) + y'(0) + R(s).$$

2nd Step: We solve the subsidiary equation **algebraically** for Y. Division by $s^2 + as + b$ and use of the so-called **transfer function**

$$\boxed{Q(s) = \frac{1}{s^2 + as + b}}$$

gives the solution

$$\boxed{Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)\,Q(s).} \quad \text{...(ii)}$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; thus Q is the quotient

$$\boxed{Q = \frac{Y}{R} = \frac{L(\text{output})}{L(\text{input})}}$$

and this explains the name of Q. Note that Q depends only on a and b, but does not depend on either r(t) or on the initial conditions.

3rd Step. We reduce (ii) (usually by partial fractions, as in calculus) to a sum of terms whose inverse can be found from the table, so that the solution $y(t) = L^{-1}(Y)$ of (i) is obtained.

Example 1:

Initial problem: Explanation of the basic steps

Solve $\quad y'' - y = t, \qquad y(0) = 1,\ y'(0) = 1.$

Solution. 1st Step.

By taking Laplace transform of LHS and RHS of $y'' - y = t$, we get the following subsidiary equation

$$s^2L(y) - sy(0) - y'(0) - L(y) = 1/s^2, \quad \text{thus } (s^2 - 1)Y = s + 1 + 1/s^2,$$

where $Y = L(y)$

2nd Step. The transfer function is $Q = 1/(s^2 - 1)$, and

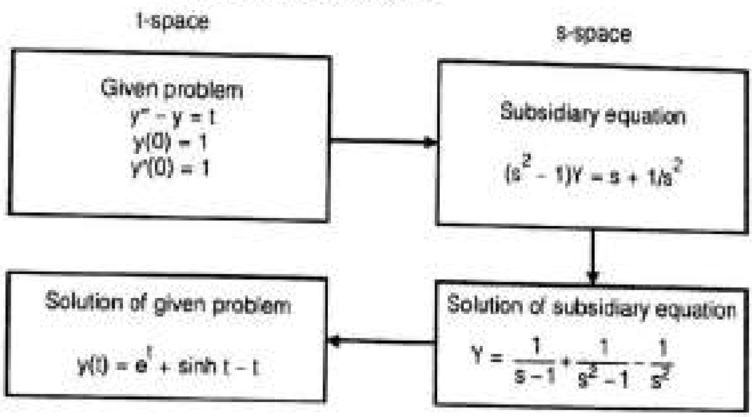
$$Y = (s+1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

3rd Step. From this expression for Y, we obtain the solution by inverse Laplace transform as follows

$$y(t) = L^{-1}(Y) = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

$$= e^t + \frac{e^t - e^{-t}}{2} - t = \frac{3e^t - e^{-t} - 2t}{2}$$

The diagram in Fig. below summarizes our approach.



Laplace transform method

Comparison with the usual method
The problem can also be solved by the usual method without using Laplace transforms as shown below:

$$y'' - y = t, \qquad y(0) = 1,\ y'(0) = 1$$
$$(D^2 - 1)y = 0$$

Auxiliary equation

$$D^2 - 1 = 0$$
$$(D+1)(D-1) = 0$$
$$m_1 = 1 \text{ and } m_2 = -1$$

So complementary function is $y = c_1 e^t + c_2 e^{-t}$

Now particular integral

$$\text{P.I.} = \frac{1}{D^2-1}(t)$$
$$= -(1 + D^2 - D^4 \cdots)t = -t + 0 - 0 \cdots = -t$$

So complete solution is

$$y = c_1 e^t + c_2 e^{-t}$$
$$y' = c_1 e^t - c_2 e^{-t}$$

Putting initial conditions y(0) = 1 and y′(0) = 1, we get

$$c_1 + c_2 = 1 \text{ and } c_1 - c_2 = 2$$

$\Rightarrow$
$$c_1 = \frac{3}{2} \text{ and } c_2 = -\frac{1}{2}$$

So C.S. is
$$y = \frac{3}{2}e^t - \frac{1}{2}e^{-t} - t = \frac{1}{2}(3e^t - e^{-t} - 2t)$$

Which is exactly the same solution as obtained by Laplace transform method.

Note: Laplace transform method has obtained the solution directly without any evaluation of constants c_1, c_2 etc.

7.4.6 Transforms of Integrals

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.14 If F(s) is the Laplace transform of function f (t), then Laplace transform of $\int_0^t f(\tau)d\tau$ is

(a) $\frac{1}{s}F(s)$
(b) $\frac{1}{s}F(s) - f(0)$
(c) $sF(s) - f(0)$
(d) $\int F(s)\,ds$

[ME, GATE-2007, 2 marks]

Solution: (a)

$$L\left[\int_0^t\int_0^t \ldots \int_0^t f(t)\,dt^n\right] = \frac{1}{s^n}F(s)$$

In this problem $n = 1$

So,
$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

7.4.7 Multiplication By t^n

If $L\{f(t)\} = \bar{f}(s)$, then $$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)], \text{ where } n = 1, 2, 3...$$

7.4.8 Division By t

If $L\{f(t)\} = \bar{f}(s)$, then $$L\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty} \bar{f}(s)ds$$

provided the integral exists.

7.5 EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS

ILLUSTRATIVE EXAMPLES

Example:

Evaluate

(a) $\int_0^{\infty} te^{-2t}\sin t\, dt$ (b) $\int_0^{\infty} \frac{\sin mt}{t}dt$ (c) $L\left\{\int_0^t \frac{e^t \sin t}{t}dt\right\}$

Solution:

(a) $$\int_0^{\infty} te^{-2t}\sin t\, dt = \int_0^{\infty} e^{-st}(t\sin t)dt \text{ where } s = 2$$

$$= L(t\sin t), \text{ by definition.}$$

$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2\times 2}{(2^2+1)^2} = \frac{4}{25}$$

(b) Since, $$L(\sin mt) = m/(s^2 + m^2) = f(s), \text{ say}$$

$$\therefore \quad L\left(\frac{\sin mt}{t}\right) = \int_s^{\infty} f(s)ds = \int_s^{\infty} \frac{m\,ds}{s^2+m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$$

Now since, $$L\left(\frac{\sin mt}{t}\right) = \int_0^{\infty} e^{-st}\frac{\sin mt}{t}dt$$

$$\therefore \quad \int_0^{\infty} e^{-st}\frac{\sin mt}{t}dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$$

Now, $$\underset{s\to 0}{\text{Lt}}\tan^{-1}(s/m) = 0 \text{ if } m > 0 \text{ or } \pi \text{ if } m < 0$$

Thus taking limits as $s \to 0$, we get

$$\int_0^{\infty} \frac{\sin mt}{t}dt = \frac{\pi}{2} \text{ if } m > 0 \text{ or } -\frac{\pi}{2} \text{ if } m < 0$$

(c) Since, $$L\left(\frac{\sin t}{t}\right) = \int_s^{\infty} \frac{ds}{s^2+1} = \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s.$$

$$\therefore \quad L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1), \text{ by shifting property}$$

Thus, $$L\left[\int_0^t\left\{e^t\left(\frac{\sin t}{t}\right)\right\}dt\right] = \frac{1}{s}\cot^{-1}(s-1)$$

Example:

Evaluate $\int_{-\infty}^{\infty} 12\cos 2\pi t \cdot \frac{\sin 4\pi t}{4\pi t} \cdot dt$

Solution:

Since function is even function so,

$$I = 2\int_{-\infty}^{\infty} 12\cos 2\pi t \cdot \frac{\sin 4\pi t}{4\pi t} \cdot dt$$

$$= \frac{3}{\pi}\int_0^{\infty}\left[\frac{\sin 6\pi t + \sin 2\pi t}{t}\right]dt \qquad \text{[Note: } 2\cos C\sin D = \sin(C+D) + \sin(C-D)\text{]}$$

$$= \frac{3}{\pi}\left[\int_0^{\infty}\frac{\sin 6\pi t}{t}dt + \int_0^{\infty}\frac{\sin 2\pi t}{t}dt\right]$$

$$= \frac{3}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 3$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.15 Evaluate $\int_0^{\infty} \frac{\sin t}{t} dt$

(a) π
(b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$
(d) $\frac{\pi}{8}$

[CE, GATE-2007, 2 marks]

Solution: (b)

Since, $L(\sin mt) = \frac{m}{(s^2 + m^2)} = f(s)$, say.

$$\therefore \quad L\left(\frac{\sin mt}{t}\right) = \int_s^{\infty} f(s)\,ds = \int_s^{\infty}\frac{m\,ds}{s^2 + m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s^{\infty}$$

or by Definition,

$$\int_0^{\infty} e^{-st}\frac{\sin mt}{t}dt = \frac{\pi}{2}\tan^{-1}\frac{s}{m}$$

Now $\underset{s\to 0}{\text{Lt}}\tan^{-1}\left(\frac{s}{m}\right) = 0$ if $m > 0$ or π if $m < 0$.

Thus taking limits as $s \to 0$, we get

$$\int_0^{\infty}\frac{\sin mt}{t}dt = \frac{\pi}{2} \text{ if } m > 0 \text{ or } -\frac{\pi}{2} \text{ if } m < 0.$$

In this problem $m = 1$ which is > 0 therefore the answer is $\frac{\pi}{2}$.

7.6 INVERSE TRANSFORMS – METHOD OF PARTIAL FRACTIONS

Having found the Laplace Transforms of a few functions, let us now determine the inverse transforms of given functions of s. We have seen that L{(f(t)} in each case, is a rational algebraic function. Hence to find the inverse transforms, we first express the given function of s into partial fractions which will, then, be recognizable as one of the following standard forms:

1. $L^{-1}\left[\frac{1}{s}\right]=1$
2. $L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$
3. $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$, n = 1, 2, 3,...
4. $L^{-1}\left[\frac{1}{(s-a)^n}\right]=\frac{e^{at}t^{n-1}}{(n-1)!}$
5. $L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at$
6. $L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$
7. $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$
8. $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\cosh at$
9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right]=\frac{1}{b}e^{at}\sin bt$
10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos bt$
11. $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]=\frac{1}{2a}t\sin at$
12. $L^{-1}\left[\frac{1}{(s^2-a^2)^2}\right]=\frac{1}{2a^2}(\sin at - at\cos at)$

All these results need to be memorised. The results (1) to (10) follow at once from their corresponding results in transforms of elementary functions and properties of Laplace transforms. Results (11) and (12) can be proved.

Note on Partial Fractions:

To resolve a given fraction into partial fractions, we first factorise the denominator into real factors. These will be either linear or quadratic, and some factors repeated. We know from algebra that a proper fraction can be resolved into a sum of partial fractions such that

1. to a non-repeated linear factor s – a in the denominator corresponds a partial fraction of the form A/(s – a).
2. to a repeated linear factor $(s-a)^r$ in the denominator corresponds the sum of r partial fractions of the form $\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\ldots+\frac{A_r}{(s-a)^r}$.
3. to a non-repeated quadratic factor (s^2+as+b) in the denominator, corresponds a partial fraction of the form $\frac{As+B}{s^2+as+b}$.
4. to a repeated quadratic factor $(s^2+as+b)^r$ in the denominator, corresponds the sum of r partial fractions of the form $\frac{A_1s+B_1}{s^2+as+b}+\frac{A_2s+B_2}{(s^2+as+b)^2}+\ldots+\frac{A_rs+B_r}{(s^2+as+b)^r}$.

Then we have to determine the unknown constants A, A_1, B_1 etc.

In all other cases, equate the given fraction to a sum of suitable partial fractions in accordance with 1 to 4 above, having found the partial fractions corresponding to the non-repeated linear factors by the above rule. Then multiply both sides by the denominator of the given fraction and equate the coefficients of like powers of s or substitute convenient numerical values of s on both sides. Finally solve the simplest of the resulting equations to find the unknown constants.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.16 The inverse Laplace transform of $\frac{1}{(s^2+s)}$ is

(a) $1 + e^t$
(b) $1 - e^t$
(c) $1 - e^{-t}$
(d) $1 + e^{-t}$

[ME, GATE-2009, 1 mark]

Solution: (c)

$$L^{-1}\left(\frac{1}{s^2+s}\right) = ?$$

$$\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}\left(\frac{1}{s^2+s}\right) = L^{-1}\left(\frac{1}{s}\right) - -L^{-1}\left(\frac{1}{s+1}\right)$$

$$= 1 - e^{-t}$$

[Using standard formulae]

Standard formula:

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

Q.17 The Laplace transform of a function f(t) is $\frac{1}{s^2(s+1)}$. The function f(t) is

(a) $t - 1 + e^{-t}$
(b) $t + 1 + e^{-t}$
(c) $-1 + e^{-t}$
(d) $2t + e^t$

[ME, GATE-2010, 2 marks]

Solution: (a)

$$f(t) = L^{-1}\left[\frac{1}{s^2(s+1)}\right]$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$\frac{1}{s^2(s+1)} = \frac{As(s+1) + B(s+1) + C(s^2)}{s^2(s+1)}$$

Matching coefficient of s^2, s and constant in numerator we get,

$$A + C = 0 \quad \text{... (i)}$$

$$A + B = 0 \quad \text{... (ii)}$$

$$B = 1 \quad \text{... (iii)}$$

Solving we get A = –1, B = 1, C = 1

So,

$$f(t) = L^{-1}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right] = -1 + t + e^{-t} = t - 1 + e^{-t}$$

Q.18 The inverse Laplace transform of the function $F(s) = \dfrac{1}{s(s+1)}$ is given by

(a) $f(t) = \sin t$ (b) $f(t) = e^{-t} \sin t$
(c) $f(t) = e^{-t}$ (d) $f(t) = 1 - e^{-t}$

[ME, GATE-2012, 2 marks]

Solution: (d)

$$F(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+B(s)}{s(s+1)}$$

$\Rightarrow A(s+1) + B(s) = 1$

Put $s = 0$

$\Rightarrow A = 1$

and $s = -1$

$\Rightarrow B = -1$

So $F(s) = \dfrac{1}{s} - \dfrac{1}{s+1}$

Now $f(t) = L^{-1}(F(s)) = e^{0t} - e^{-t}$

$f(t) = 1 - e^{-t}$

Q.19 Given $\mathcal{L}^{-1}\left[\dfrac{3s+1}{s^2 + 4s^2 + (K-3)s}\right]$. If $\lim\limits_{t \to \infty} f(t) = 1$, then the value of K is

(a) 1 (b) 2
(c) 3 (d) 4

[EC, GATE-2010, 2 marks]

Solution: (d)

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Given that, $F(s) = \left[\dfrac{3s+1}{s^3 + 4s^2 + (K-3)s}\right]$

$$\lim_{t \to \infty} f(t) = 1$$

$$\Rightarrow \lim_{s \to 0} s\left[\frac{3s+1}{s^3 + 4s^2 + (K-3)s}\right] = 1$$

$$\Rightarrow \lim_{s \to 0}\left[\frac{3s+1}{s^2 + 4s + (K-3)}\right] = 1$$

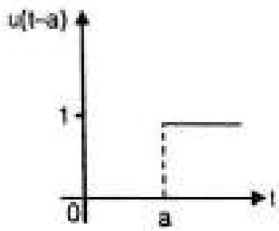
$$\Rightarrow \frac{1}{K-3} = 1$$

$$\Rightarrow K - 3 = 1$$

$$\Rightarrow K = 4$$

7.7 UNIT STEP FUNCTION

At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the unit step function (or Heaviside's unit function*).



Def. The unit step function $u(t-a)$ is defined as follows

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

where a is always positive.

7.7.1 Transform of Unit Function

$$L[(u(t-a)] = \int_0^{\infty} e^{-st} u(t-a)dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^{\infty} e^{-st} \cdot 1\, dt = 0 + \left|\frac{e^{-st}}{-s}\right|_a^{\infty}$$

Thus, $L|u(t-a)| = e^{-as/s}$.

The product $f(t)\, u(t-a) = \begin{cases} 0 & \text{for } t < 0 \\ f(t) & \text{for } t \geq a \end{cases}$

The function f(t – a) . u(t – a) represents the graph f(t) shifted through a distance a to the right and is of special importance.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.20 A delayed unit step function is defined as $u(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases}$. Its Laplace transform is

(a) $a.e^{-as}$

(b) $\frac{e^{-as}}{s}$

(c) $\frac{e^{as}}{s}$

(d) $\frac{e^{as}}{a}$

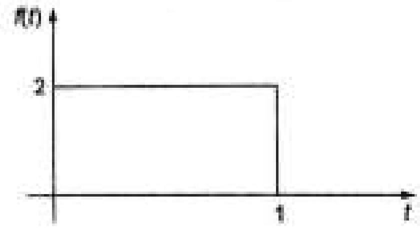
[ME, GATE-2004, 2 marks]

Solution: (d)

$$L[U(t-a)] = \int_0^{\infty} e^{-St} U(t-a)dt$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_0^{\infty} e^{-st} \cdot 1 \cdot dt = 0 + \int_a^{\infty} e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^{\infty} = \frac{e^{-as}}{s}$$

Q.21 Laplace transform of the function $f(t)$ is given by $F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$. Laplace transform of the function shown below is given by



(a) $\frac{1-e^{-2s}}{s}$

(b) $\frac{1-e^{-s}}{2s}$

(c) $\frac{2-2e^{-s}}{s}$

(d) $\frac{1-2e^{-s}}{s}$

[ME, GATE-2015 : 2 Marks, Set-3]

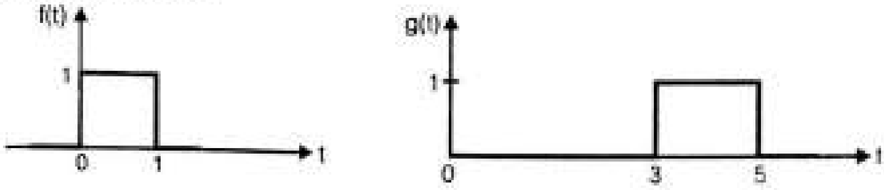
Solution: (c)

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt = \int_0^1 2e^{-st}dt + \int_1^{\infty} 0\cdot e^{-st}dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{2}{-s}[e^{-s}-1] = \frac{2(1-e^{-s})}{s} = \frac{2-2e^{-s}}{s}$$

Common Data Questions 22 and 23

Given f(t) and g(t) as shown below:



Q.22 g(t) can be expressed as

(a) $g(t) = f(2t-3)$

(b) $g(t) = f\left(\frac{t}{2}-3\right)$

(c) $g(t) = f\left(2t-\frac{3}{2}\right)$

(d) $g(t) = f\left(\frac{t}{2}-\frac{3}{2}\right)$

[EE, GATE-2010, 2 marks]

Solution: (d)

We need $g(3) = f(0)$ and $g(5) = f(1)$

Only choice (d) satisfies both these conditions as seen below:

Choice (d) is $g(t) = f\left(\frac{t}{2}-\frac{3}{2}\right)$

$$g(3) = f\left(\frac{3}{2}-\frac{3}{2}\right) = f(0)$$

and $$g(5) = f\left(\frac{5}{2}-\frac{3}{2}\right) = f(1)$$

Q.23 The Laplace transform of g(t) is

(a) $\frac{1}{s}\left(e^{3s}-e^{5s}\right)$

(b) $\frac{1}{s}\left(e^{-5s}-e^{-3s}\right)$

(c) $\frac{e^{-3s}}{s}\left(1-e^{-2s}\right)$

(d) $\frac{1}{s}\left(e^{5s}-e^{3s}\right)$

[EE, GATE-2010, 2 marks]

Solution: (c)

By definition of Laplace transform,

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$L[f(t)] = \int_0^3 e^{-st} f(t)\,dt + \int_3^5 e^{-st} f(t)\,dt + \int_5^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^3 e^{-st}.0.dt + \int_3^5 e^{-st}.1.dt + \int_5^{\infty} e^{-st}.0.dt$$

$$= \left[-\frac{e^{-st}}{s}\right]_3^5 = -\left[\frac{e^{-5s}-e^{-3s}}{s}\right] = \frac{e^{-3s}-e^{-5s}}{s} = \frac{e^{-3s}}{s}\,[1-e^{-2s}]$$

7.8 SECOND SHIFTING PROPERTY

If $$L\{f(t)\} = \bar{f}(s), \text{ then}$$

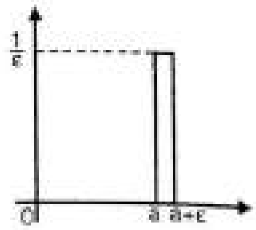
$$L[f(t-a) \times u(t-a)] = e^{-as}\bar{f}(s)$$

Proof:
$$L[f(t-a) \times u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)dt \qquad [\text{Put } t-a=u]$$

$$= \int_0^\infty e^{-s(u+a)} f(u)du = e^{-sa}\int_0^\infty e^{-su} f(u)du = e^{-as}\bar{f}(s)$$

7.9 UNIT IMPULSE FUNCTION



The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the unit impulse function (also called Dirac delta function).

Thus unit impulse function is considered as the limiting form of the function (Fig. above)

$$\delta_\varepsilon(t-a) = 1/\varepsilon, \qquad a \le t \le a+\varepsilon$$
$$= 0, \qquad \text{otherwise}$$

as $\varepsilon \to 0$. It is clear from figure that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows:

$$\delta(t-a) = \infty \qquad \text{for } t = a$$
$$= 0 \qquad \text{for } t \ne a$$

such that
$$\int_0^\infty \delta(t-a)dt = 1 \qquad (a \ge 0)$$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a+\varepsilon$. Thus

$$w(x) = w_0/\varepsilon \qquad a < x < a+\varepsilon,$$
$$= 0 \qquad \text{otherwise}$$

i.e.
$$w(x) = w_0\delta(x+a)$$

7.9.1 Transform of Unit Impulse Function

If f(t) be a function of t continuous at t = 1, then

$$\int_0^\infty f(t)\delta_0(t-a)\cdot dt = \int_a^{a+\varepsilon} f(t)\cdot\frac{1}{\varepsilon}dt = (a+\varepsilon-a)f(\eta)\cdot\frac{1}{\varepsilon} = f(\eta) \text{ where } a<\eta<a+\varepsilon.$$

by Mean value theorem for integrals.

As $\varepsilon \to 0$, we get
$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

In particular, putting $f(t) = e^{-st}$ in above integral

we have
$$\int_0^\infty e^{-st}\,\delta(t-a)dt = e^{-as}$$

Now LHS is nothing but $L[\delta(t-a)]$

$$\therefore \qquad L[\delta(t-a)] = e^{-as}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.24 A solution for the differential equation $\dot{x}(t) + 2x(t) = \delta(t)$ with initial condition $x(0-) = 0$ is

(a) $e^{-2t}u(t)$
(b) $e^{2t}u(t)$
(c) $e^{-t}u(t)$
(d) $e^{t}u(t)$

[EC, GATE-2006, 1 mark]

Solution: (a)

$$\dot{x}(t) + 2x(t) = \delta(t)$$

Taking L.T. on both sides

$$sX(s) - x(0) + 2X(s) = 1$$

$$X(s)[s+2] = 1$$

$$X(s) = \frac{1}{s+2}$$

$$x(t) = e^{-2t}u(t)$$

Q.25 Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \delta(t) \text{ with}$$

$$y(t)\Big|_{t=0} = -2 \text{ and } \frac{dy}{dt}\Big|_{t=0} = 0.$$

The numerical value of $\frac{dy}{dt}\Big|_{t=0}$ is

(a) -2
(b) -1
(c) 0
(d) 1

[EC, IN GATE-2012, 2 marks]

Solution: (d)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = \delta(t)$$

taking Laplace transform on both the sides we have

$$s^2Y(s) + 2s + 2sY(s) + 4 + Y(s) = 1$$

$$(s^2 + 2s + 1)Y(s) = -(2s + 3)$$

$$Y(s) = \frac{-(2s+3)}{(s+1)^2}$$

$$Y(s) = -\left[\frac{2}{(s+1)} + \frac{1}{(s+1)^2}\right]$$

$$\Rightarrow \quad Y(t) = -[2e^{-t} + te^{-t}]u(t)$$

$$\frac{dy}{dt} = -[-2e^{-t} + e^{-t} - te^{-t}]u(t)$$

$$\frac{dy}{dt}\Big|_{at\ t=0+} = -[-2 + 1 - 0]$$

$$\frac{dy}{dt}\Big|_{at\ t=0+} = 1$$

Q.26 Consider a causal LTI system characterized by differential equation $\frac{dy(t)}{dt}+\frac{1}{6}y(t)=3x(t)$. The response of the system to the input $x(t)=3e^{-t/3}u(t)$, where $u(t)$ denotes the unit step function, is

(a) $9e^{-t/3}u(t)$ (b) $9e^{-t/6}u(t)$

(c) $9e^{-t/3}u(t)-6e^{-t/6}u(t)$ (d) $54e^{-t/6}u(t)-54e^{-t/3}u(t)$

[EE, 2016 : 1 Mark, Set-2]

Solution: (d)

The differential equation,

$$\frac{dy(t)}{dt}+\frac{1}{6}y(t) = 3x(t)$$

So, $$sY(s)+\frac{1}{6}Y(s) = 3X(s)$$

$$Y(s) = \frac{3X(s)}{\left(s+\frac{1}{6}\right)}$$

$$X(s) = \frac{9}{\left(s+\frac{1}{3}\right)}$$

So, $$Y(s) = \frac{9}{\left(s+\frac{1}{3}\right)\left(s+\frac{1}{6}\right)} = \frac{54}{\left(s+\frac{1}{6}\right)}-\frac{54}{\left(s+\frac{1}{3}\right)}$$

So, $$y(t) = (54e^{-1/6t}-54e^{-1/3t})u(t)$$

7.10 PERIODIC FUNCTIONS

If f(t) is a periodic function with period T, i.e. f(t + T) = f(t), then

$$L\{f(t)\} = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$$

Example:

If $f(t)=\begin{cases}\sin t & 0<t<\pi\\ 0 & \pi<t<2\pi\end{cases}$, f(t) is periodic function with time period 2π. Determine the Laplace transform of f(t).

Solution:

Laplace transform of periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt = \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}}{s^2+1}(-s\cdot\sin t-1\cos t)\right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi}}\left[\frac{e^{-\pi s}}{s^2+1}+\frac{1}{s^2+1}\right] = \frac{(1+e^{-\pi s})}{(s^2+1)(1-e^{-\pi s})(1+e^{-\pi s})} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

OOOO

CHAPTER 8

Fourier Series

INTRODUCTION

Fourier series is an approximation process where any general (periodic or aperiodic) signal is expressed as sum of harmonically related sinusoids. It gives us frequency domain representation.

If the signal is periodic Fourier series represents the signal in the entire interval $(-\infty, \infty)$, i.e. Fourier series can be generalized for periodic signals only.

Definition

Suppose f is a piecewise continuous periodic function of period $2L$, then f has a Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Where the coefficients a's and b's are given by the Euler-Fourier formulas:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

8.1 DRICHILET'S CONDITIONS

The sufficient condition for the convergence of a Fourier series are called Drichilet's conditions.

1. $f(x)$ is periodic, single valued and finite.
2. $f(x)$ has a finite number of finite discontinuities in any one period.
3. $f(x)$ has a finite number of maxima and minima.

8.1.1 Fourier Cosine and Sine Series

If f is an even periodic function of period $2L$, then its Fourier series contains only cosine (include, possibly, the constant term) terms. It will not have any sine term. That is, its Fourier series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Its Fourier coefficients are determined by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3, \ldots$$

If f is an odd periodic function of period $2L$, then its Fourier series contains only sine terms. It will not have any cosine term. That is, its Fourier series is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Its Fourier coefficients are determined by

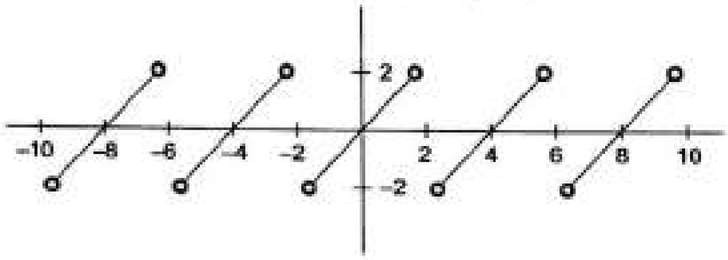
$$a_n = 0, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \qquad n = 0, 1, 2, 3, \ldots$$

ILLUSTRATIVE EXAMPLES

Example:

Find a Fourier series for $f(x) = x$, $-2 < x < 2$, $f(x+4) = f(x)$



Solution:

First note that $T = 2L = 4$, hence $L = 2$

The constant term is one half of a_0,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{2}\int_{-2}^{2} x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_{-2}^{2} = \frac{1}{2}(2-2) = 0$$

The rest of the cosine coefficients, for $n = 1, 2, 3, \ldots$, are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{2}\int_{-2}^{2} x\cos\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{2} - \frac{2}{n\pi}\int_{-2}^{2}\sin\frac{n\pi x}{2}dx\right)$$

$$= \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_{-2}^{2}\right)$$

$$= \frac{1}{2}\left[\left(0 + \frac{4}{n^2\pi^2}\cos(n\pi)\right) - \left(0 + \frac{4}{n^2\pi^2}\cos(-n\pi)\right)\right] = 0$$

Hence, there is no non-zero cosine coefficient for this function. That is, its Fourier series contains no cosine terms at all. (We shall see the significance of this fact a little later).

The sine coefficients, for $n = 1, 2, 3, \ldots$, are

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{2}\int_{-2}^{2} x\sin\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{2} - \frac{-2}{n\pi}\int_{-2}^{2}\cos\frac{n\pi x}{2}dx\right)$$

$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_{-2}^{2}\right)$$

$$= \frac{1}{2}\left[\left(\frac{-4}{n\pi}\cos(n\pi) - 0\right) - \left(\frac{4}{n\pi}\cos(-n\pi) - 0\right)\right]$$

$$= \frac{-2}{n\pi}[(\cos(n\pi) + \cos(n\pi)] = \frac{-4}{n\pi}\cos(n\pi)$$

$$= \begin{cases}\frac{4}{n\pi}, & n = \text{odd} \\ \frac{-4}{n\pi}, & n = \text{even}\end{cases} = \frac{(1)^{n+1}4}{n\pi}$$

Therefore,
$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

Example:

Find a Fourier series for $f(x) = x$, $0 < x < 4$, $f(x + 4) = f(x)$. How will it be different from the series in the previous example?

Solution:

$$a_0 = \frac{1}{2}\int_0^4 x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^4 = \frac{1}{4}(8 - 0) = 4$$

For $n = 1, 2, 3, \ldots$
$$a_n = \frac{1}{2}\int_0^4 x\cos\frac{n\pi x}{2}dx = \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left[\left(0 + \frac{4}{n^2\pi^2}\cos(2n\pi)\right) - \left(0 + \frac{4}{n^2\pi^2}\cos(0)\right)\right] = 0$$

$$b_n = \frac{1}{2}\int_0^4 x\sin\frac{n\pi x}{2}dx = \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left[\left(\frac{-8}{n\pi}\cos(2n\pi - 0) - (0 - 0)\right)\right] = \frac{-4}{n\pi}$$

Consequently,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = 2 + \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

Comment: Just because a Fourier series could have infinitely many (non-zero) terms does not mean that it will always have that many terms. If a periodic function f can be expressed by finitely many terms normally found in a Fourier series, then the expression must be the Fourier series of f. (This is analogous to the fact that the Maclaurin's series of any polynomial function is just the polynomial itself, which is a sum of finitely many powers of x.)

Example: The Fourier series (period 2π) representing

$$f(x) = 5 + \cos(4x) - \sin(5x) \text{ is just } f(x) = 5 + \cos(4x) - \sin(5x).$$

Example: The Fourier series (period 2π) representing $f(x) = 6\cos(x)\sin(x)$ is not exactly itself as given, since the product $\cos(x)$ is not a term in a Fourier series representation. However, we can use the double-angle formula of sine to obtain the result: $6\cos(x)\sin(x) = 3\sin(2x)$. Consequently, the Fourier series is $f(x) = 3\sin(2x)$.

8.1.2 The Cosine and Sine Series Extensions

If f and f' are piecewise continuous functions defined on the interval $0 \le t \le L$, then f can be extended into an even periodic function, F, of period $2L$, such that $f(x) = F(x)$ on the interval $[0, L]$, and whose Fourier series is, therefore, a cosine series. Similarly, f can be extended into an odd periodic function of period $2L$, such that $f(x) = F(x)$ on the interval $(0, L)$, and whose Fourier series is, therefore, a sine series. The process that such extensions are obtained is often called cosine.sine half-range expansions.

Even (cosine series) extension of $f(x)$

Given $f(x)$ defined on $[0, L]$. Its even extension of period $2L$ is

$$F(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L < x < 0 \end{cases} \qquad F(x + 2L) = F(x)$$

Where,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad \text{such that}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 3, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3, \ldots$$

Odd (sine series) extension of $f(x)$

Given $f(x)$ defined on $[0, L]$. Its odd extension of period $2L$ is

$$F(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0, L \\ -f(-x), & -L < x < 0 \end{cases} \qquad F(x + 2L) = F(x)$$

Where,

$$F(x) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{L}, \qquad \text{such that}$$

$$a_n = 0, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 3, \ldots$$

ILLUSTRATIVE EXAMPLES

Example:

Let $f(x) = x$, $0 \le x < 2$. Find its cosine and sine series extensions of period 4.

Solution:

Cosine series: $$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cos\frac{(2n-1)\pi x}{2}$$

Sine series: $$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 The Fourier series of the function,

$f(x) = 0, \quad -\pi < x \le 0$

$= \pi - x, \quad 0 < x < \pi$

in the interval $[-\pi, \pi]$ is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^3} + \dots\right] + \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right]$$

The convergence of the above Fourier series at $x = 0$ gives

(a) $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(c) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(d) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

[CE, 2016 : 1 Mark, Set-II]

Solution: (c)

The function is $f(x) = 0$,

$$-p < x \le 0$$
$$= p - x,\ 0 < x < \pi$$

And Fourier series is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right] + \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right] \quad \dots(i)$$

At $x = 0$, (a point of discontinuity), the fourier series converges to $\frac{1}{2}\left[f(0^{-1}) + f(0^{+})\right]$,

where $f(0^-) = \lim_{x\to 0}(\pi - x) = \pi$

Hence, eq. (i), we get

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \dots\right]$$

$$\Rightarrow \quad \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \frac{\pi^2}{8}$$

OOOO

CHAPTER 9

Second Order Linear Partial Differential Equations

INTRODUCTION

We are about to study a simple type of partial differential equations (PDEs): the second order linear PDEs. Recall that a partial differential equation is any differential equation that contains two or more independent variables. Therefore the derivative(s) in the equation are partial derivatives. We will examine the simplest case of equations with 2 independent variables. A few second order linear PDEs in 2 variables are:

$a^2 u_{xx} = u_t$ (one-dimensional heat conduction equation)

$a^2 u_{xx} = u_{tt}$ (one-dimensional wave equation)

$u_{xx} + u_{yy} = 0$ (two-dimensional heat conduction equation)

9.1 CLASSIFICATION OF SECOND ORDER LINEAR PDEs

Consider the general form of a second order linear partial differential equation in 2 variables with constant coefficients:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + f_u = g(x, y)$$

For the equation to be of second order, a, b, and c cannot all be zero. Definen its discriminant to be $b^2 - 4ac$. The properties and behaviour of its solution are largely dependent of its type, as classified below.

If $b^2 - 4ac > 0$, then the equation is called **hyperbolic**. The wave equation is one such example.

If $b^2 - 4ac = 0$, then the equation is called **parabolic**. The heat conduction equation is one such example.

If $b^2 - 4ac < 0$, then the equation is called **elliptic**. The Laplace equation is one such example.

Example:

Consider the one-dimensional damped wave equation $9u_{xx} = u_{tt} + 6u_t$.

Solution:

It can be rewriten as: $9u_{xx} - u_{tt} - 6u_t = 0$. It has coefficients $a = 9$, $b = 0$, and $c = -1$. Its discriminant is $9 > 0$. Therefore, the equation is hyperbolic.

9.2 UNDAMPED ONE-DIMENSIONAL WAVE EQUATION: VIBRATIONS OF AN ELASTIC STRING

Consider a piece of thin flexible string of length L, of negligible weight. Suppose the two ends of the string are firmly secured ("clamped") at some supports so they will not move. Assume the set-up has no damping. Then, the vertical displacement of the string, $0 < x < L$, and at any time $t > 0$, is given by the displacement function $u(x, t)$. It satisfies the homogeneous one-dimensional undamped wave equation:

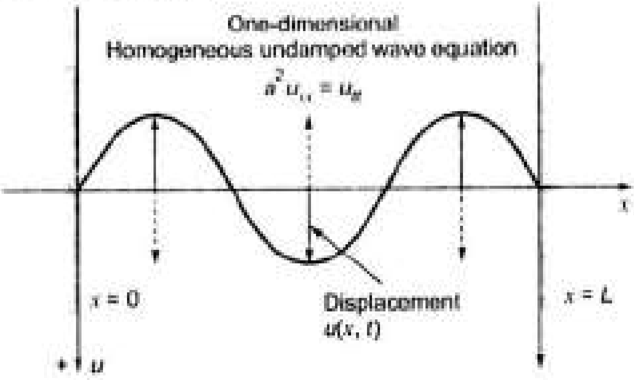
$$a^2 u_{xx} = u_{tt}$$

Where the constant coefficient a^2 is given by the formula $a^2 = T/\rho$, such that a = horizontal propagation speed (also known as phase velocity) of the wave motion, T = force of tension exerted on the string, ρ = Mass density (mass per unit length). It is subjected to the homogeneous boundary conditions.

$$u(0, t) = 0, \text{ and } u(L, t) = 0, \; t > 0$$

The two boundary conditions reflect that the two ends of the string are clamped in fixed positions. Therefore, they are held motionless at all time.

The equation comes with 2 initial conditions, due to the fact that it contains the second partial derivative term u_{tt}. The two initial conditions are the $u_t(x, 0)$, both are arbitrary functions of x alone. (Note that the string is vibrates, vertically, in place. The resulting wave form, or the wave-like "shape" of the string, is what moves horizontally.)



Hence, what we have is the following initial-boundary value problem:

(wave equation)	$a^2 u_{xx} = u_{tt}$,	$0 < x < L, t > 0$
(Boundary conditions)	$u(0, t) = 0$, and	$u(L, t) = 0$,
(Initial conditions)	$u(x, 0) = f(x)$, and	$u_t(x, 0) = g(x)$

We first let $u(x, t) = X(x)\, T(t)$ and separate the wave equation into two ordinary differential equations. Substituting $u_{xx} = X''T$ and $u_{tt} = XT''$ into the wave equation, it becomes

$$a^2 X''T = XT''$$

Dividing both sides by $a^2 XT$:

$$\frac{X''}{X} = \frac{T''}{a^2T}$$

As for the heat conduction equation, it is customary to consider the constant a^2 as a function of t and group it with the rest of t-terms. Insert the constant of separation and break apart the equation:

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$$

$$\frac{X''}{X} = -\lambda \quad \Rightarrow \quad X'' = -\lambda X \quad \Rightarrow \quad X'' + \lambda X = 0$$

$$\frac{T''}{a^2T} = -\lambda \quad \Rightarrow \quad T'' = -a^2\lambda T \quad \Rightarrow T'' + a^2\lambda T = 0$$

The boundary conditions also separate:

$$u(0, t) = 0 \Rightarrow X(0)\,T(t) = 0 \Rightarrow X(0) = 0 \quad \text{or} \quad T(t) = 0$$
$$u(L, t) = 0 \Rightarrow X(L)\,T(t) = 0 \Rightarrow X(L) = 0 \quad \text{or} \quad T(t) = 0$$

As usual, in order to obtain nontrivial solutions, we need to choose $X(0) = 0$ and $X(L) = 0$ as the new boundary conditions. The result, after separation of variables, is the following simultaneous system of ordinary differential equations, with a set of boundary conditions:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$
$$T'' + a^2 \lambda T = 0$$

The next step is to solve the eigen value problem:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

The solutions are given by taking λ negative

Eigen values: $\lambda = \dfrac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots$

Eigen functions: $X_n = \sin\dfrac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$

Next, substitute the eigen values found above into the second equation to find $T(t)$. After putting eigen values λ into it, the equation of T becomes

$$T'' + a^2\frac{n^2\pi^2}{L^2}T = 0$$

It is a second order homogeneous linear equation with constant coefficients. It's characteristic have a pair of purely imaginary complex conjugate roots:

$$r = \pm\frac{an\pi}{L}i$$

Thus, the solutions are simple harmonic:

$$T_n(t) = A_n\cos\frac{an\pi t}{L} + B_n\sin\frac{an\pi t}{L}, \quad n = 1, 2, 3, \ldots$$

Multiplying each pair of X_n and T_n together and sum them up, we find the general solution of the one-dimensional wave equation, with both ends fixed, to be

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{an\pi t}{L} + B_n\sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

There are two sets of (infinitely many) arbitrary coefficients. We can solve for them using the two initial conditions.

Set $t = 0$ and apply the first initial condition, the initial (vertical) displacement of the string $u(x, 0) = f(x)$, we have

$$u(x, 0) = \sum_{n=1}^{\infty}(A_n\cos(0) + B_n\sin(0))\sin\frac{n\pi x}{L}$$
$$= \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L} = f(x)$$

Therefore, we see that the initial displacement $f(x)$ needs to be a Fourier sine series. Since $f(x)$ can be an arbitrary function, this usually means that we need to expand it into its odd periodic extension (of period $2L$). the coefficients A_n are then found by the relation $A_n = b_n$, where b_n are the corresponding Fourier sine coefficients of $f(x)$. That is

$$A_n = b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Notice that the entire sequence of the coefficients A_n are determined exactly by the initial displacement. They are completely independent of the other sequence B_n, which are determined solely by the second initial condition, the initial (vertical) velocity of the string. To find B_n, we differentiate $u(x, t)$ with respect to t apply the initial velocity, $u_t(x, 0) = g(x)$.

$$u_t(x, t) = \sum_{n=1}^{\infty}\left(-A_n\frac{an\pi}{L}\sin\frac{an\pi t}{L} + B_n\frac{an\pi}{L}\cos\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

Set $t = 0$ and equate it with $g(x)$:

$$u_t(x, 0) = \sum_{n=1}^{\infty}B_n\frac{an\pi}{L}\sin\frac{n\pi x}{L} = g(x)$$

We see that $g(x)$ needs also be a Fourier sine series. Expand it into its odd periodic extension (period $2L$), if necessary. Once $g(x)$ is written into a sine series, the previous equation becomes

$$u_t(x, 0) = \sum_{n=1}^{\infty}B_n\frac{an\pi}{L}\sin\frac{n\pi x}{L} = g(x) = \sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

Compare the coefficients of the like sine terms, we see

$$B_n\frac{an\pi}{L} = b_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

Therefore,

$$B_n = \frac{L}{an\pi}b_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

As we have seen, half of the particular solution is determined by the initial displacement, the other half by the initial velocity. The two halves are determined independent of each other. Hence, if the initial displacement $f(x) = 0$, then all $A_n = 0$ and $u(x, t)$ contains no sine-terms of t. If the initial velocity $g(x) = 0$, then all $B_n = 0$ and $u(x, t)$ contains no cosine-terms of t.

Let us take another look and summarize the result for these 2 easy special cases, when either $f(x)$ or $g(x)$ is zero.

Special case I: Non-zero initial displacement, zero initial velocity: $f(x) \neq 0$, $g(x) = 0$.
Since $g(x) = 0$, then $B_n = 0$ for all n.

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \qquad n = 1, 2, 3, \ldots$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty}A_n\cos\frac{an\pi t}{L}\sin\frac{n\pi x}{L}$$

ILLUSTRATIVE EXAMPLES

Example:

Solve the one-dimensional wave problem.

$$9u_{xx} = u_{tt}, \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(5, t) = 0,$$
$$u(x, t) = 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x),$$
$$u_t(x, 0) = 0.$$

Solution:

First note that $a^2 = 9$ (so, $a = 3$), and $L = 5$

The general solution is, therefore,

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5}\right)\sin\frac{n\pi x}{5}$$

Since $g(x) = 0$, it must be that all $B_n = 0$. We just need to find A_n. We also see that $u(x, 0) = f(x)$ is already in the form of a Fourier sine series. Therefore, we just need to extract the corresponding Fourier sine coefficients:

$$A_5 = b_5 = 4,$$
$$A_{10} = b_{10} = -1,$$
$$A_{25} = b_{25} = -3,$$
$$A_n = b_n = 0, \text{ for all other } n,\ n \neq 5, 10, \text{ or } 25.$$

Hence, the particular solution is

$$u(x, t) = 4\cos(3\pi t)\sin(\pi x) - \cos(6\pi t)\sin(2\pi x) - 3\cos(15\pi t)\sin(5\pi x)$$

Example:

Solve the one-dimensional wave problem.

$$9u_{xx} = u_{tt}, \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(5, t) = 0,$$
$$u(x, 0) = 0$$
$$u_t(x, 0) = 4.$$

Solution:

As in the previous example, $a^2 = 9$ (so, $a = 3$), and $L = 5$

Therefore, the general solution remains

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5}\right)\sin\frac{n\pi x}{5}$$

Now, $f(x) = 0$, consequently all $A_n = 0$. We just need to find B_n. The initial velocity $g(x) = 4$ is a constant function. It is not an odd periodic function. Therefore, we need to expand it into its odd periodic extension (period $T = 10$), then equate it with $u_t(x, 0)$. In short:

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx = \frac{2}{3n\pi}\int_0^5 4\sin\frac{n\pi x}{5}dx$$

$$= \begin{cases} \dfrac{80}{3n^2\pi^2}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty}\frac{80}{3(2n-1)^2\pi^2}\sin\frac{3(2n-1)\pi t}{5}\sin\frac{(2n-1)\pi x}{5}$$

9.2.1 Summary of Wave Equation: Vibrating String Problems

The vertical displacement of a vibrating string of length L, securely clamped at both ends, of negligible weight and without damping, is described by the homogeneous undamped wave equation initial-boundary value problem:

$$a^2 u_{xx} = u_{tt} \qquad 0 < x < L, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(L, t) = 0,$$
$$u(x, 0) = f(x), \quad \text{and} \quad u_t(x, 0) = g(x)$$

The general solution is

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

The particular solution can be found by the formulas:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \text{ and}$$

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

The solution waveform has a constant (Horizontal) propagation speed, in both directions of the x-axis, of a. The vibrating motion has a (vertical) velocity given by $u_t(x, t)$ at any location $0 < x < L$ along the string.

Exercise:

1. Solve the vibrating string problem of the given initial conditions.

$$4\,u_{xx} = u_{tt} \quad 0 < x < \pi, \quad t > 0,$$
$$u(0, t) = 0, \quad u(\pi, t) = 0,$$

   (a) $u(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x)$; $u_t(x, 0) = 0.$

   (b) $u(x, 0) = 0$; $u_t(x, 0) = 6$

   (c) $u(x, 0) = 0$; $u_t(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x)$

2. Solve the vibrating string problem.

$$100\,u_{xx} = u_{tt} \quad 0 < x < 2, \quad t > 0,$$
$$u(0, t) = 0, \quad u(2, t) = 0,$$
$$u(x, 0) = 32\sin(\pi x) + e^2\sin(3\pi x) + 25\sin(6\pi x).$$
$$u_t(x, 0) = 6\sin(2\pi x) - 16\sin(5\pi x/2)$$

3. Solve the vibrating string problem.

$$25\,u_{xx} = u_{tt} \quad 0 < x < 1, \quad t > 0,$$
$$u(0, t) = 0 \text{ and } u(2, t) = 0,$$
$$u(x, 0) = x - x^2,$$
$$u_t(x, 0) = \pi$$

4. Verify that the D'Alembert solution, $u(x, t) = [F(x - at) + F(x + at)]/2$, where $F(x)$ is an odd periodic function of period $2L$ such that $F(x) = f(x)$ on the interval $0 < x < L$, indeed satisfies the initial-boundary value problem by checking that it satisfies the wave equation, boundary conditions, and initial conditions.

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0,$$
$$u(0, t) = 0, \qquad u(L, t) = 0,$$
$$u(x, 0) = f(x), \qquad u_t(x, 0) = 0.$$

5. Use the method of separation of variables to solve the following wave equation problem where the string is rigid, but not fixed in place, at both ends (i.e., it is inflexible at the end points such that the slope of displacement curve is always zero at both ends, but the two ends of the string are allowed to freely slide in the vertical direction).

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0,$$
$$u_x(0, t) = 0, \quad u_x(L, t) = 0,$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x).$$

6. What is the steady-state displacement of the string in #5? What is $\lim_{t\to\infty} u(x, t)$? Are they the same?

Answers:

1. (a) $u(x, t) = 12\cos(4t)\sin(2x) - 16\cos(10t)\sin(5x) + 24\cos(12t)\sin(6x)$.
   (c) $u(x, t) = 3\sin(4t)\sin(2x) - 1.6\sin(10t)\sin(5x) + 24\sin(12t)\sin(6x)$.

5. (a) The general solution is $u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L}\right)\cos\frac{n\pi x}{L}$

   The particular solution can be found by the formulas:

   $$A_0 = \frac{1}{L}\int_0^L f(x)\,dx,\ A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ B_0 = \frac{1}{L}\int_0^L g(x)\,dx,\text{ and } B_n = \frac{2}{an\pi}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx$$

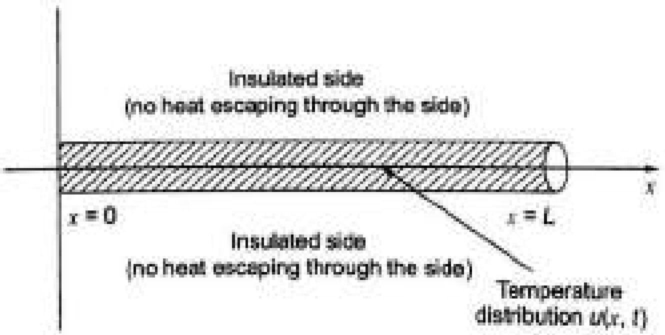
6. The steady-state displacement is the constant term of the solution, A_0. The limit does not exist unless $u(x, t) = C$ is a constant function, which happens when $f(x) = C$ and $g(x) = 0$, in which case the limit is C. They are not the same otherwise.

9.3 THE ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

Consider a thin bar of length L, of uniform cross-section and constructed of homogeneous material. Suppose that the side of the bar is perfectly insulated so no heat transfer could occur through it (heat could possibly still move into or out of the bar through the two ends of the bar). Thus, the movement of heat inside the bar could occur only in the x-direction. then, the amount of heat content at any place inside the bar, $0 < x < L$, and at any time $t > 0$, is given by the temperature distribution function $u(x, t)$. It satisfies the homogeneous one-dimensional heat conduction equation:

$$a^2 u_{xx} = u_t$$

Where the constant coefficient α^2 is the thermo diffusivity of the bar, given by $a^2 = k/\rho s$. (k = thermal conductivity, ρ = density, s = specific heat, of the material of the bar.)



Further, let us assume that both ends of the bar are kept constantly at 0 degree temperature.

(Heat conduction equation) $a^2 u_{xx} = u_t$, $0 < x < L$, $t > 0$,
(Boundary conditions) $u(0, t) = 0$, and $u(L, t) = 0$,
(Initial condition) $u(x, 0) = f(x)$

9.3.1 Conduction Problem

The general solution of the initial-boundary value problem given by the one-dimensional heat conduction modeling a bar that has both of its ends at 0 degree. The general solution is

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 n^2 \pi^2 t / L^2} \sin\frac{n\pi x}{L}$$

Setting $t = 0$ and applying the initial condition $u(x, 0) = f(x)$, we get

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x)$$

We know that the above equation says that the initial condition needs to be an odd periodic function of period $2L$. Since the initial condition could be an arbitrary function, it usually means that we would need to "force the issue" and expand it into an odd periodic function of period $2L$. That is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Therefore, the particular solution is found by setting all the coefficients $C_n = b_n$, where b_n's are the Fourier sine coefficients of (or the odd periodic extension of) the initial condition $f(x)$:

$$C_n = b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

ILLUSTRATIVE EXAMPLES

Example:

Solve the heat conduction problem.

$$8 u_{xx} = u_t, \quad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0 \text{ and } u(5, t) = 0,$$
$$u(x, 0) = 2\sin(\pi x) - 4\sin(2\pi x) - \sin(5\pi x)$$

Solution:

Since the standard form of the heat conduction equation $a^2 u_{xx} = u_t$, we see that $a^2 = 8$; and we also note that $L = 5$. Therefore, the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 n^2 \pi^2 t / L^2} \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} C_n e^{-8n^2 \pi^2 t / 25} \sin\frac{n\pi x}{5}$$

The initial condition, $f(x)$, is already an odd periodic function (notice that it is a Fourier sine series) of the correct period $T = 2L = 10$.

Therefore, no additional calculation is needed, and all we need to do is to extract the correct Fourier sine coefficients from $f(x)$. To wit

$$C_5 = b_5 = 2,$$
$$C_{10} = b_{10} = -4,$$
$$C_{25} = b_{25} = 1,$$

$$C_n = b_n = 0, \text{ for all other } n,\ n \neq 5,\ 10, \text{ or } 25.$$

Hence,

$$u(x, t) = 2e^{-8(5^2)\pi^2 t/25}\sin(\pi x) - 4e^{-8(10)^2\pi^2 t/25}\sin(2\pi x) + e^{-8(25^2)\pi^2 t/25}\sin(5\pi x)$$

Example:

What will the particular solution be if the initial condition is $u(x, 0) = x$ instead? That is, solve the following heat conduction problem:

$$8\,u_{xx} = u_t, \quad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0 \text{ and } u(5, t) = 0,$$
$$u(x, 0) = x$$

Solution:

The general solution is still

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-8n^2\pi^2 t/25}\sin\frac{n\pi x}{5}$$

The initial condition is an odd function, but it is not a periodic function. Therefore, it needs to be expanded into its odd periodic extension of period $10(T = 2L)$. Its coefficients are, for $n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{5}\int_0^5 x\sin\frac{n\pi x}{5}dx$$

$$= \frac{2}{5}\left(\frac{-5x}{n\pi}\cos\frac{n\pi x}{5}\Big|_0^5 - \frac{-5}{n\pi}\int_0^5 \cos\frac{n\pi x}{5}dx\right)$$

$$= \frac{2}{5}\left(\frac{-5x}{n\pi}\cos\frac{n\pi x}{5} + \frac{25}{n^2\pi^2}\sin\frac{n\pi x}{5}\Big|_0^5\right)$$

$$= \frac{2}{5}\left[\left(\frac{-25}{n\pi}\cos(n\pi) - 0\right) - (0 - 0)\right] = \frac{-10}{n\pi}\cos(n\pi)$$

$$= \begin{cases} \dfrac{10}{n\pi}, & n = \text{odd} \\ \dfrac{-10}{n\pi}, & n = \text{even} \end{cases} = \frac{(-1)^{n+1}10}{n\pi}$$

The resulting sine series is (representing the function $f(x) = x$, $-5 < x < 5$, $f(x + 10) = f(x)$):

$$f(x) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{5}$$

The particular solution can then be found by setting each coefficient, C_n, to be the corresponding Fourier sine coefficient of the series above, $C_n = b_n = \dfrac{(-1)^{n+1}(10)}{n\pi}$. Therefore, the particular solution is

$$u(x, t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-8n^2\pi^2 t/25}\sin\frac{n\pi x}{5}$$

The Steady-State Solution

The steady-state solution, $v(x)$, of a heat conduction problem is the part of the temperature distribution function that is independent of time t. It represents the equilibrium temperature distribution. To find it, we note the fact that it is a function of x alone, yet it has to satisfy the heat conduction equation. Since $v_{xx} = v''$ and $v_t = 0$, substituting them into the heat conduction equation, we get,

$$a^2 v_{xx} = 0$$

Divide both sides by a^2 and integrate twice with respect to x, we find that $v(x)$ must be in the form of a degree 1 polynomial:

$$v(x) = Ax + B$$

Then, rewrite the boundary conditions in terms of v: $u(0, t) = v(0) = T_1$, and $u(L, t) = v(L) = T_2$. Apply those 2 conditions to find that:

$$v(0) = T_1 = A(0) + B = B \quad \Rightarrow \quad B = T_1$$
$$v(L) = T_2 = AL + B = AL + T_1 \quad \Rightarrow \quad A = (T_2 - T_1)/L$$

Therefore, $$v(x) = \frac{T_2 - T_1}{L}x + T_1$$

Further examples of steady-state solutions of the heat conduction equation:

ILLUSTRATIVE EXAMPLES

Example:

Find $v(x)$, given each set of boundary conditions below:

1. $u(0, t) = 50, \quad u_x(6, t) = 0$
2. $u(0, t) - 4u_x(0, t) = 0, \quad u_x(10, t) = 25$

Solution:

1. We are looking for a function of the form $v(x) = Ax + B$ that satisfies the given boundary conditions. Its derivative is then $v'(x) = A$. The two boundary conditions can be rewritten to be $u(0, t) = v(0) = 50$, and $u_x(6, t) = v'(6) = 0$. Hence,

$$v(0) = 50 = A(0) + B = B \quad \Rightarrow \quad B = 50$$
$$v'(0) = 0 = A \quad \Rightarrow \quad A = 0$$

Therefore, $v(x) = 0x + 50 = 50$

2. The two boundary conditions can be rewritten be $v(0) - 4v'(0) = 0$, and $v'(10) = 25$.

Hence, $$v(0) - 4v'(0) = 0 = (A(0) + B) - 4A = -4A + B$$
$$4v'(10) = 25 = A \quad \Rightarrow \quad A = 25$$

Substitute $A = 25$ into the first equation: $0 = -4A + B = -100 + B$

$$\Rightarrow \quad B = 100$$

Therefore, $v(x) = 25x + 100$.

9.4 LAPLACE EQUATION FOR A RECTANGULAR REGION

Consider a rectangular of length a and width b. Suppose the top, bottom, and left sides border free-space; while beyond the right side there lies a source of heat/gravity/magnetic flux, whose strength is given by $f(y)$. The potential function at any point (x, y) within this rectangular region, $u(x, y)$, is then described by the boundary value problem:

(2-dim. Laplace equation) $\quad u_{xx} + u_{yy} = 0, \quad 0 < x < a,\ 0 < y < b,$

(Boundary conditions) $\quad u(x, 0) = 0$, and $\quad u(x, b) = 0,$

$u(0, y) = 0$, and $\quad u(a, b) = f(y).$

The separation of variables proceeds similarly. A slight difference here is that $Y(y)$ is used in the place of $T(t)$. Let $u(x, y) = X(x)\, Y(y)$ and substituting $u_{xx} = XY''$ into the wave equation, it becomes

$$XY'' + XY'' = 0,$$
$$X''Y = -XY''$$

Dividing both sides by XY:

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Now that the independent variables are separated to the two sides, we can insert the constant of separation. Unlike the previous instances, it is more convenient to denote the constant as positive λ instead.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\frac{X''}{X} = \lambda \quad \Rightarrow \quad X'' = \lambda X \quad \Rightarrow \quad X'' - \lambda X = 0$$

$$-\frac{Y''}{Y} = \lambda \quad \Rightarrow \quad Y'' = \lambda Y \quad \Rightarrow \quad Y'' + \lambda Y = 0$$

The boundary conditions also separate:

$$u(x, 0) = 0 \;\Rightarrow\; X(x)\,Y(0) = 0 \;\Rightarrow\; X(x) = 0 \text{ or } Y(0) = 0$$
$$u(x, b) = 0 \;\Rightarrow\; X(x)\,Y(b) = 0 \;\Rightarrow\; X(x) = 0 \text{ or } Y(b) = 0$$
$$u(0, y) = 0 \;\Rightarrow\; X(0)\,Y(y) = 0 \;\Rightarrow\; X(0) = 0 \text{ or } Y(y) = 0$$
$$u(a, y) = f(y) \;\Rightarrow\; X(a)\,Y(y) = f(y) \;\Rightarrow\; \text{[cannot be simplified further]}$$

$$X'' - \lambda X = 0, \qquad X(0) = 0,$$
$$Y'' - \lambda Y = 0, \qquad Y(0) = 0 \text{ and } Y(b) = 0$$

Plus the fourth boundary condition, $u(a, y) = f(y)$

The next step is to solve the eigen value problem. Notice that there is another slight difference. Namely that this time it is the equation of Y that gives rise to the two-point boundary value problem which we need to solve.

$$Y'' + \lambda Y = 0, \qquad Y(0) = 0, \quad Y(b) = 0$$

However, except for the fact that the variables is y and the function is Y, rather than x and X, respectively, we have already seen this problem before (more than once, as a matter of fact ; here the constant $L = b$). The eigen values of this problem are

$$\lambda = \sigma^2 = \frac{n^2\pi^2}{b^2}, \qquad n = 1, 2, 3, \ldots$$

Their corresponding eigen function are

$$Y_n = \sin\frac{n\pi y}{b}, \qquad n = 1, 2, 3, \ldots$$

Once we have found the eigen values, substitute I into the equation of x. We have the equation, together with one boundary condition:

$$X'' - \frac{n^2\pi^2}{b^2}X = 0, \qquad X(0) = 0.$$

Its characteristic equation, $r^2 - \frac{n^2\pi^2}{b^2} = 0$, has real roots $r = \pm\frac{n\pi}{b}$.

Hence, the general solution for the equation of x is

$$X = C_1 e^{\frac{n\pi}{b}x} + C_2 e^{\frac{-n\pi}{b}x}$$

The single boundary condition gives

$$X(0) = 0 = C_1 + C_2 \Rightarrow C_2 = C_1$$

Therefore, for $n = 1, 2, 3, \ldots$

$$X_n = C_n\left(e^{\frac{n\pi}{b}x} - e^{\frac{-n\pi}{b}x}\right)$$

Because of the identity for the hyperbolic sine function

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2},$$

the previous expression is often rewritten in terms of hyperbolic sine:

$$X_n = K_n \sinh\frac{n\pi x}{b}, \quad n = 1, 2, 3, \ldots$$

The coefficients satisfy the relation: $K_n = 2C_n$

Combining the solutions of the two equations, we get the set of solutions that satisfies the two-dimensional Laplace equation, given the specified boundary conditions:

$$u_n(x, y) = X_n(x)Y_n(y) = K_n \sin\frac{n\pi x}{b}\sin\frac{n\pi y}{b}, \quad n = 1, 2, 3, \ldots$$

$$u(x, y) = \sum_{n=1}^{\infty} K_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

This solution, of course, is specific to the set of boundary conditions

$u(x, 0) = 0$, and $u(x, b) = 0$,

$u(0, y) = 0$ and $u(a, y) = f(y)$

To find the particular solution, we will use the fourth boundary condition, namely, $u(a, y) = f(y)$.

$$u(a, y) = \sum_{n=1}^{\infty} K_n \sinh\frac{an\pi}{b}\sin\frac{n\pi y}{b} = f(y)$$

We have seen this story before. There is nothing really new here. the summation above is a sine series whose Fourier sine coefficients are $b_n = K_n \sin(an\pi/b)$. Therefore, the above relation says that the last boundary condition, $f(y)$, must either be an odd periodic function (period = $2b$), or it needs to be expanded into one. Once we have $f(y)$ as a Fourier sine series, the coefficients K_n of the particular solution can then be computed:

$$K_n \sinh\frac{an\pi}{b} = b_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

Therefore,

$$K_n = \frac{b_n}{\sinh\frac{an\pi}{b}} = \frac{2}{b\sinh\frac{an\pi}{b}}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 The solution of the partial differential equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ is of the form

(a) $C\cos(kt)\left|C_1 e^{(\sqrt{k/\alpha})} + C_2 e^{-(\sqrt{k/\alpha})x}\right|$

(b) $Ce^{kt}\left[C_1 e^{(\sqrt{k/\alpha})x} + C_2 e^{-(\sqrt{k/\alpha})x}\right]$

(c) $Ce^{kt}\left[C_1 \cos(\sqrt{k/\alpha})x + C_2 \sin(-\sqrt{k/\alpha})x\right]$

(d) $C\sin(kt)\left[C_1 \cos(\sqrt{k/\alpha})x + C_2 \sin-(\sqrt{k/\alpha})x\right]$

[CE, 2016 : 1 Mark, Set-I]

Solution: (b)

The PDE $$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

Solution of (i) is

$$u(x, t) = (A\cos px + B\sin px)Ce^{-p^2\alpha t}$$

Put $-p^2\alpha = k$

$$\Rightarrow \quad p = \sqrt{-\frac{k}{\alpha}} = \sqrt{\frac{k}{\alpha}}\, i$$

Putting value of p in eq. (i)

$$u(x, t) = \left(A\cos\sqrt{\frac{k}{\alpha}}x + b\sinh\sqrt{\frac{k}{\alpha}}x\right)Ce^{kt}$$

$$= Ce^{kt}\left[A\left\{\frac{e^{\sqrt{\frac{k}{\alpha}}x} + e^{-\sqrt{\frac{k}{\alpha}}x}}{2}\right\} + B\left\{\frac{e^{\sqrt{\frac{k}{\alpha}}x} - e^{-\sqrt{\frac{k}{\alpha}}x}}{2}\right\}\right]$$

$$= Ce^{kt}\left[e^{\sqrt{\frac{k}{\alpha}}x}\left\{\frac{A+B}{2}\right\} + e^{-\sqrt{\frac{k}{\alpha}}x}\left\{\frac{A-B}{2}\right\}\right]$$

$$= Ce^{kt}\left[c_1 e^{\sqrt{\frac{k}{\alpha}}x} + c_2 e^{-\sqrt{\frac{k}{\alpha}}x}\right]$$

Q.2 The type of partial differential equation $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + 3\frac{\partial^2 P}{\partial x \partial y} + 2\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} = 0$ is

(a) elliptic
(b) parabolic
(c) hyperbolic
(d) none of these

[CE, 2016 : 1 Mark, Set-I]

Solution: (c)

Comparing the given equation with the general form of second order partial differential equation, we have A = 1, B = 3, C = 1 ⇒ $B^2 - 4AC = 5 > 0$

∴ PDE is Hyperbola.

OOOO

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